

Rational $O(2)$ –Equivariant Spectra

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Abstract

We find a simple algebraic model for rational $O(2)$ –equivariant spectra, via a series of Quillen equivalences. This model, along with an Adams short exact sequence, will allow us to easily perform constructions and calculations. Furthermore all of our constructions are monoidal, so we can use this classification to understand ring spectra and module spectra via the algebraic model.

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1 Introduction

Equivariant cohomology theories are a fundamental tool for studying spaces with a G -action. To study these cohomology theories, it is helpful to understand the G -spectra that represent them. The homotopy category of G -spectra is particularly complicated. It contains all the information of the stable homotopy category, as well as equivariant information such as the Burnside ring of G , the real representation ring of G and the group cohomology of G . A standard and fruitful method to make this category easier to study is to rationalise. Since the rational stable homotopy category is equivalent to the category of graded rational vector spaces, we have removed this source of complexity. However, much of the interesting behaviour that comes from the group is preserved and made tractable.

Thus we want to understand the rational G -equivariant stable homotopy category. By understand, we mean that we have an abelian category $\mathcal{A}(G)$, equipped with a model structure and a Quillen equivalence between the model category of $\mathcal{A}(G)$ and the model category of rational G -spectra. A Quillen equivalence gives us an equivalence of homotopy categories and also tell us that all further homotopical structures,

such as homotopy limits or Toda brackets are preserved by this equivalence. The algebraic model should be explicit and manageable so that constructing objects or maps is straightforward. Furthermore the Quillen equivalences provide us with an Adams spectral sequence relating maps in the homotopy category of $\mathcal{A}(G)$ to maps in the homotopy category of rational spectra.

These aims have been completed for finite groups, through the work of [GM95, Appendix A] and [SS03b, Example 5.1.2]. These results were improved to the level of a monoidal Quillen equivalence in [Bar09a]. The case of a torus has also been extensively studied in [Gre99], [Shi02] and more recently in [GS11]. The next logical case to study is extensions of a torus by a finite group, the canonical example being $O(2)$. The homotopy level classification appeared in [Gre98b], which describes the algebraic model for $O(2)$ as the product of two split pieces. The first, called the cyclic part, is the algebraic model for $SO(2)$ along with a skewed action of $W = O(2)/SO(2)$. The second, called the dihedral part, behaves much more like the case of a finite group. The dihedral part is much simpler to understand than the cyclic part and has been shown to be Quillen equivalent to a specific localisation of rational $O(2)$ -spectra in the preprint [Bar08a]. The aim of this paper is to complete this classification and make sure that it respects the monoidal product. We will then be able to use this model to understand ring spectra or modules over ring spectra via the algebraic model.

The algebraic model for the cyclic part is defined in section 3. We use the notation $\partial\mathcal{A}(\mathcal{C})_{qd}$ for this model category. We use $\mathcal{A}(\mathcal{D})$ to denote the algebraic model for the dihedral part, as defined in section 6. These are well-behaved monoidal model categories which are relatively easy to define. In particular, it is very easy to perform constructions and calculations within these models.

Our main theorem is given below and is the combination of theorem 2.3.1, proposition 2.3.2, theorem 5.2.1 and theorem 6.2.13.

Theorem *The model category of rational $O(2)$ -equivariant spectra is Quillen equivalent to the algebraic model $A(O(2)) = \partial\mathcal{A}(\mathcal{C})_{qd} \times \mathcal{A}(\mathcal{D})$. Furthermore these Quillen equivalences are all symmetric monoidal, hence the homotopy categories of rational $O(2)$ -equivariant spectra and $A(O(2))$ are equivalent as symmetric monoidal categories.*

1.1 Highlights

Of particular use for general stable homotopy theorists is section 5. Here we study right Bousfield localisations of stable model categories. As usual we assume that the model category is right proper and cellular (a mildly stronger version of cofibrantly generated) and that we are localising at a set of cofibrant objects K . If we then ask that the collection of K -cellular objects (all those objects built by homotopy colimits from objects of K) is closed under desuspension in the homotopy category, then we

see that the right Bousfield localisation is also stable and cellular. Furthermore we are able to explicitly describe the generating cofibrations of this localisation. This normally requires the much more restrictive condition that all objects of the original model category be fibrant.

If we further assume that our model category is monoidal and that the smash product of any two elements of K is also K -cellular, then we can conclude that the right localisation is also a monoidal model category. This taming of the usually mysterious right Bousfield localisation greatly increases our ability to study such localisations, which are fundamental to homotopy theory.

A major difficulty in the study of rational $SO(2)$ -spectra (and hence $O(2)$ -spectra) has been the lack of a monoidal model category on the algebraic model of [Gre99]. We are able to create such a model structure via the methods of [BR11]. This is another example of a model category without enough projectives, which has enough dualisable objects. Roughly speaking, the dualisable objects are those A , such that $A \otimes -$ has a left adjoint. That this model category is monoidal follows from the fact that the monoidal product of two dualisable objects is again dualisable. We discuss how these methods may be extended to the case of a torus in remark 3.5.7.

The cyclic part of this paper is a specialisation of [GS11] to the case of $SO(2)$. Many simplifications occur and so this paper makes a good introduction to the general case. We are also able to upgrade their classification of $SO(2)$ to one that is compatible with the smash product of spectra.

This paper also represents the prototype for other extensions of a torus by a finite group. We expect the methods and ideas used here to be replicated by any further study. Indeed, if we let G be an extension of $SO(2)$ by a finite group, then by our methods we can classify the ‘cyclic’ part of rational G -spectra in terms of an algebraic model. We discuss this further in remark 4.6.3.

1.2 Organisation

We begin by describing the model category of rational $O(2)$ -spectra in section 2. We then split the category into the category of cyclic spectra and dihedral spectra. Since the cyclic case is the hardest, we start with that half. Our first task is to describe the algebraic model and give it a monoidal model category, we do so in section 3. Then we show that we have cyclic spectra and the algebraic model we make are Quillen equivalent in section 4. Section 5 describes how these Quillen equivalences are all compatible with the monoidal product. With the harder half completed, we perform the monoidal classification of dihedral spectra in section 6.

2 Rational $O(2)$ –Spectra

In this section we introduce the model category we wish to study and show that it splits into the product of two localisations. We will also need some basic results on the group $O(2)$ as well.

2.1 The group $O(2)$

We will use the notation D_{2n}^h to represent the dihedral subgroup of order $2n$ containing h , an element of $O(2) \setminus SO(2)$. The closed subgroups of $O(2)$ are $O(2)$, $SO(2)$, the finite dihedral groups D_{2n}^h for each h and the cyclic groups C_n ($n \geq 1$). We write W for the group of order 2, since it is the Weyl group of $SO(2)$ in $O(2)$ $W = O(2)/SO(2)$. Let $H \leq O(2)$, then $N_{O(2)}(H)$ is the normaliser in $O(2)$ of H , it is the largest subgroup of $O(2)$ in which H is normal. The Weyl-group of H in $O(2)$ is $W_{O(2)}(H) := N_{O(2)}(H)/H$. The normaliser of D_{2n}^h in $O(2)$ is D_{4n}^h , thus the Weyl group of D_{2n}^h is W . The cyclic groups are normal, hence the Weyl group of C_n is $O(2)/C_n \cong O(2)$. The Weyl group of $SO(2)$ is again W and the Weyl group of $O(2)$ is the trivial group.

Recall the following material from [LMSM86, Chapter V, Section 2]. Define $\mathcal{FO}(2)$ to be the set of those subgroups of $O(2)$ with finite index in their normaliser (or equally, with finite Weyl-group), equipped with the Hausdorff topology. This is an $O(2)$ –space via the conjugation action of $O(2)$ on its subgroups. This space is of interest due to tom Dieck’s ring isomorphism:

$$A(O(2)) \otimes \mathbb{Q} := [S, S]^{O(2)} \otimes \mathbb{Q} \xrightarrow{\cong} C(\mathcal{FO}(2)/O(2), \mathbb{Q})$$

where $C(\mathcal{FO}(2)/O(2), \mathbb{Q})$ is the ring of continuous maps from $\mathcal{FO}(2)/O(2)$ to \mathbb{Q} , considered as a discrete space. We draw $\mathcal{FO}(2)/O(2)$ below as Figure 1. We will sometimes write D_{2n} for (D_{2n}^h) , the conjugacy class of D_{2n}^h . The point $O(2)$ is a limit point of this space.

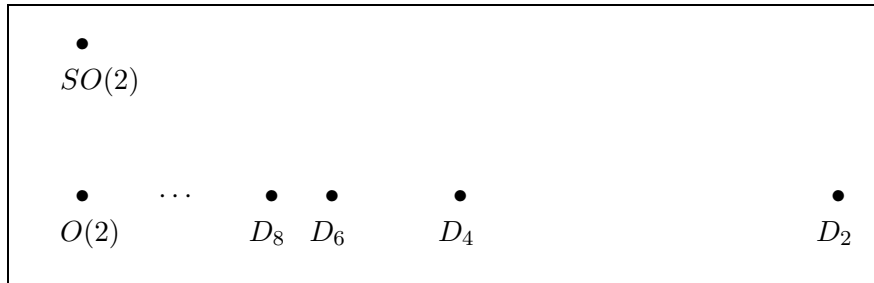


Figure 1: $\mathcal{FO}(2)/O(2)$.

Definition 2.1.1 Define \mathcal{C} to be the set consisting of the cyclic groups and $SO(2)$, this is a family of subgroups (that is, \mathcal{C} is closed under conjugation and subgroups). Let \mathcal{D} be the complement of \mathcal{C} in the set of all (closed) subgroups of $O(2)$.

Lemma 2.1.2 We define idempotents of $C(\mathcal{FO}(2)/O(2), \mathbb{Q})$ as follows: $e_{\mathcal{C}}$ is the characteristic function of $SO(2)$, $e_{\mathcal{D}} = e_{\mathcal{C}} - 1$ and e_n is the characteristic function of D_{2n} for each $n \geq 1$. We also let $f_n = e_{\mathcal{D}} - \sum_{k=1}^{n-1} e_k$.

Remark 2.1.3 Since Hasse squares are important to the cyclic part, we note here that there is a Hasse square describing $A(O(2))$. That is, we can describe it as part of a pullback of commutative rings. We use the notation $\mathbb{Q}\langle e \rangle$ to mean of copy of the ring \mathbb{Q} with basis element e .

$$\begin{array}{ccc} A(O(2)) & \longrightarrow & \mathbb{Q}\langle e_{\mathcal{C}} \rangle \oplus \mathbb{Q}\langle e_{O(2)} \rangle \\ \downarrow & & \downarrow \\ \prod_{k \geq 1} \mathbb{Q}\langle e_k \rangle & \longrightarrow & \operatorname{colim}_n \prod_{k \geq n} \mathbb{Q}\langle e_k \rangle \end{array}$$

The right hand vertical sends $e_{\mathcal{C}}$ to zero and on $\mathbb{Q}\langle e_{O(2)} \rangle$ it is induced by the maps $\mathbb{Q}\langle e_{O(2)} \rangle \rightarrow \mathbb{Q}\langle e_k \rangle$ which send $e_{O(2)}$ to e_k . The bottom horizontal map is the canonical map into a colimit.

An explicit isomorphism from $C(\mathcal{FO}(2)/O(2), \mathbb{Q})$ to this pullback square is easy to construct. Take a continuous map $f: \mathcal{FO}(2)/O(2) \rightarrow \mathbb{Q}$, we then send this to $(f(D_{2k}))_k \in \prod_{k \geq 1} \mathbb{Q}\langle e_k \rangle$ and $(f(SO(2)), f(O(2))) \in \mathbb{Q}\langle e_{\mathcal{C}} \rangle \oplus \mathbb{Q}\langle e_{O(2)} \rangle$. Continuity of f guarantees that we have an element of the pullback.

Lemma 2.1.4 Let D_{2n}^h be a dihedral subgroup of $O(2)$ of order $2n$. Then for each $k \mid n$ the rational Burnside ring of D_{2n}^h has idempotents e_{C_k} and $e_{D_{2k}}$. The collection of idempotents e_{C_k} and $e_{D_{2k}}$ for $k \mid n$ gives a maximal orthogonal decomposition of the identity.

The inclusion map $D_{2n}^h \rightarrow O(2)$ induces the following map $A(O(2)) \rightarrow A(D_{2n}^h)$

$$\begin{array}{lll} e_{\mathcal{C}} & \mapsto & \sum_{k \mid n} e_{C_k} \\ e_{\mathcal{D}} & \mapsto & \sum_{k \mid n} e_{D_{2k}} \\ e_k & \mapsto & e_{D_{2k}} \quad k \mid n \\ e_k & \mapsto & 0 \quad k \nmid n. \end{array}$$

2.2 Model categories for spectra

We need a monoidal model category of $O(2)$ -equivariant spectra. For technical reasons we will need to use equivariant EKMM S -modules for ‘dihedral’ part of the classification. Conversely, for the ‘cyclic’ part, we will use equivariant orthogonal spectra as it is a simpler category to work with. Both these categories are introduced in [MM02].

We note here that we use the term M-cellular model category for what is more usually just known as a cellular model category, see [Hir03, Definition 12.1.1]. This is because we want to talk of ‘cellularised model categories’, that is to say model categories that have been right Bousfield localised.

We also remark that what we call a monoidal Quillen pair in this paper is sometimes known a weak monoidal Quillen pair. This is a Quillen adjunction (L, R) between monoidal model categories, such that R is a monoidal functor and L is ‘monoidal up to weak equivalence’. This means that the canonical map $L(X \otimes Y) \rightarrow LX \wedge LY$ is a weak equivalence for cofibrant X and Y and we also require a condition on the units, see [SS03a, Definition 3.6.]. It follows that the left adjoint of such an adjunction is strong monoidal on the homotopy category. We use the qualifier strong to indicate when a monoidal Quillen pair has a left adjoint L such that the canonical map $L(X \otimes Y) \rightarrow LX \wedge LY$ is a natural isomorphism for any X and Y .

Definition 2.2.1 *Let $O(2)\mathcal{IS}$ be the category of $O(2)$ -equivariant orthogonal spectra and let $O(2)\mathcal{MS}$ be the category of $O(2)$ -equivariant EKMM S -modules. In each case we are working with a complete $O(2)$ -universe \mathcal{U} .*

Each category has a monoidal, proper, M-cellular, stable model structure where weak equivalences are those maps f such that $\pi_*^H(f)$ is an isomorphism for all closed subgroups H of $O(2)$. In the case of EKMM S -modules we are using the generalised cellular model structure. For this model structure, the generating cofibrations are defined in terms of the spectra $\Sigma_V^\infty O(2)/H_+ \wedge S^n$ for H a closed subgroup of $O(2)$ and V any finite dimensional sub- $O(2)$ -inner product space of \mathcal{U} .

These two model categories of spectra are Quillen equivalent, though one has to pass through the positive model structure on $O(2)$ -equivariant orthogonal spectra (which we denote by $O(2)\mathcal{IS}_+$), see [MM02, Section III.5]. It follows that we are free to choose either model category, as they have the same homotopy theory.

Proposition 2.2.2 *There are symmetric monoidal Quillen equivalences*

$$\begin{aligned} \mathbb{N} : O(2)\mathcal{IS}_+ &\rightleftarrows O(2)\mathcal{MS} : \mathbb{N}^\# \\ \text{Id} : O(2)\mathcal{IS}_+ &\rightleftarrows O(2)\mathcal{IS} : \text{Id}. \end{aligned}$$

Following [Bar09b, Section 5] and using [MM02, Theorem IV.6.3], we localise both of these model categories at a rational sphere spectrum $S^0\mathbb{Q}$. This process is a Bousfield localisation, hence the category is unchanged, but the model structure has been altered. We call the weak equivalences of the localised model structure **rational equivalences**. A map f is a rational equivalence in these localised model structures if $\pi_*^H(f) \otimes \mathbb{Q}$ is an isomorphism for all closed subgroups H of $O(2)$. We call these model structures the **rational model structures**.

Definition 2.2.3 *Let $O(2)\mathcal{IS}_{\mathbb{Q}}$ be the category of $O(2)$ -equivariant orthogonal spectra equipped with the rational model structure. Similarly, $O(2)\mathcal{MS}_{\mathbb{Q}}$ is the category of $O(2)$ -equivariant EKMM S -modules with the rational model structure.*

Hence we have two Quillen equivalent model categories of rational $O(2)$ -equivariant spectra. These categories are still proper, M-cellular and monoidal. We want to understand the homotopy theory contained in these model categories by finding some simpler algebraic category that has the same homotopy theory.

2.3 Splitting rational $O(2)$ -spectra

We know by [Bar09b, Section 6] and [Gre98b] that the homotopy theory of rational $O(2)$ -spectra splits into two pieces. We work with the EKMM S -module category for the moment, though the result will hold equally well for equivariant orthogonal spectra.

Using the idempotent $e_{\mathcal{C}}$ we can make spectra $e_{\mathcal{C}}S$ and $(1 - e_{\mathcal{C}})S = e_{\mathcal{D}}S$. We can then Bousfield localise the model category of rational $O(2)$ -spectra at these objects to obtain $L_{e_{\mathcal{C}}S}O(2)\mathcal{MS}_{\mathbb{Q}}$ and $L_{e_{\mathcal{D}}S}O(2)\mathcal{MS}_{\mathbb{Q}}$. The weak equivalences of $L_{e_{\mathcal{C}}S}O(2)\mathcal{MS}_{\mathbb{Q}}$ are those maps f such that $e_{\mathcal{C}}S \wedge f$ is a rational equivalence. one can calculate and show that this is equivalent to asking that $e_{\mathcal{C}}\pi_*^H(f) \otimes \mathbb{Q}$ is an isomorphism for all closed subgroups H . The analogous statements hold when $e_{\mathcal{C}}$ is replaced by $e_{\mathcal{D}}$. The result below is [Bar09b, Corollary 6.3].

Theorem 2.3.1 *The adjunction below is a strong monoidal Quillen equivalence.*

$$\Delta : O(2)\mathcal{MS}_{\mathbb{Q}} \rightleftarrows L_{e_{\mathcal{C}}S}O(2)\mathcal{MS}_{\mathbb{Q}} \times L_{e_{\mathcal{D}}S}O(2)\mathcal{MS}_{\mathbb{Q}} : \Pi$$

We can identify these localised homotopy categories more clearly. Note that for any X and Y in $O(2)\mathcal{MS}_{\mathbb{Q}}$ $[X, Y]_{\mathbb{Q}}^{O(2)}$ is a module over the ring $[S, S]_{\mathbb{Q}}^{O(2)}$. Hence we have the following isomorphisms of sets of maps in the homotopy category.

$$\begin{aligned} [X, Y]_{\mathbb{Q}}^{O(2)} &\cong e_{\mathcal{C}}[X, Y]_{\mathbb{Q}}^{O(2)} \times e_{\mathcal{D}}[X, Y]_{\mathbb{Q}}^{O(2)} \\ e_{\mathcal{C}}[X, Y]_{\mathbb{Q}}^{O(2)} &\cong \mathrm{Ho} L_{e_{\mathcal{C}}S}O(2)\mathcal{MS}_{\mathbb{Q}}(X, Y) \\ e_{\mathcal{D}}[X, Y]_{\mathbb{Q}}^{O(2)} &\cong \mathrm{Ho} L_{e_{\mathcal{D}}S}O(2)\mathcal{MS}_{\mathbb{Q}}(X, Y) \end{aligned}$$

We will find it useful to have versions of cyclic spectra and dihedral spectra defined in terms of modules over ring spectra. We can think of this as making the model category more complicated in return for a simpler model structure.

There are commutative S -algebras $S_{\mathcal{C}}$ and $S_{\mathcal{D}}$ (in $O(2)\mathcal{MS}$) given by localising the sphere spectrum at the objects $e_{\mathcal{C}}S \wedge S^0\mathbb{Q}$ and $e_{\mathcal{D}}S \wedge S^0\mathbb{Q}$, see [EKMM97, Theorem VIII.2.2]. Thus we know the following

$$\pi_*^H(S_{\mathcal{C}}) = e_{\mathcal{C}}\pi_*^H(S) \otimes \mathbb{Q} \quad \pi_*^H(S_{\mathcal{D}}) = e_{\mathcal{D}}\pi_*^H(S) \otimes \mathbb{Q}.$$

The model structure on $S_{\mathcal{C}}$ and $S_{\mathcal{D}}$ -modules that we use has fibrations and weak equivalences defined in terms of underlying EKMM S -modules.

Proposition 2.3.2 *There are strong monoidal Quillen equivalences as below, with U denoting the forgetful functors.*

$$\begin{aligned} S_{\mathcal{C}} \wedge - : L_{e_{\mathcal{C}}} SO(2) \mathcal{M} S_{\mathbb{Q}} &\xrightleftharpoons{\sim} S_{\mathcal{C}}\text{-mod} : U_{\mathcal{C}} \\ S_{\mathcal{D}} \wedge - : L_{e_{\mathcal{D}}} SO(2) \mathcal{M} S_{\mathbb{Q}} &\xrightleftharpoons{\sim} S_{\mathcal{D}}\text{-mod} : U_{\mathcal{D}} \\ \mathbb{N} : \mathbb{N}^{\#} S_{\mathcal{C}}\text{-mod} &\xrightleftharpoons{\sim} S_{\mathcal{C}}\text{-mod} : \mathbb{N}^{\#} \end{aligned}$$

We write $[-, -]^{S_{\mathcal{C}}}$ and $[-, -]^{S_{\mathcal{D}}}$ to denote maps in the homotopy categories of $S_{\mathcal{C}}$ -modules and $S_{\mathcal{D}}$ -modules. We use $[-, -]_*^{S_{\mathcal{C}}}$ to denote the graded set of maps. Note that $[X, Y]^{S_{\mathcal{C}}} = e_{\mathcal{C}}[X, Y]^{O(2)}$, where we have omitted the forgetful functor from $S_{\mathcal{C}}$ -modules to $O(2)$ -spectra. The analogous statement also holds for $S_{\mathcal{D}}$.

We now choose the precise model categories that we will use in the later sections.

Definition 2.3.3 *The category $\mathbb{N}^{\#} S_{\mathcal{C}}\text{-mod}$ is called the model category of **cyclic** $O(2)$ -spectra. We call $S_{\mathcal{D}}\text{-mod}$ the model category of **dihedral** $O(2)$ -spectra.*

From now on, for notational purposes, we will often use \mathbb{T} for the group $SO(2)$.

3 The algebraic model $\mathcal{A}(\mathcal{C})$

Before we start studying cyclic spectra in more detail, we introduce the algebraic model for cyclic spectra, $\mathcal{A}(\mathcal{C})$. Our task for this section is twofold. First we must adapt the algebraic model for rational \mathbb{T} -spectra, denoted $\mathcal{A}(\mathbb{T})$, to cyclic spectra. Secondly we must construct some monoidal model structures for these algebraic models. We find it practical to begin with the case of the circle group and then extend this to the case of cyclic spectra. Hence we spend some time describing $\mathcal{A}(\mathbb{T})$

3.1 The model $\mathcal{A}(\mathbb{T})$

The algebraic category for rational \mathbb{T} -equivariant cohomology theories is established in [Gre99]. We introduce that category, explain how to turn this into a differential graded category and then define the injective model structure.

Definition 3.1.1 *Let \mathcal{F} be the set of finite subgroups of \mathbb{T} . Let $\mathcal{O}_{\mathcal{F}}$ be the graded ring of operations $\prod_{H \in \mathcal{F}} \mathbb{Q}[c_H]$ with c_H of degree -2 . Let e_H be the idempotent arising from projection onto factor H . In general, let ϕ be a subset of \mathcal{F} and define e_{ϕ} to be the idempotent coming from projection onto the factors in ϕ .*

We let c be the sum of all elements c_H for varying H . We can then write $c_H = e_H c$.

Definition 3.1.2 Let V be a complex representation of \mathbb{T} such that $V^{\mathbb{T}} = 0$. We call V an **admissible representation**. Let n be the complex dimension of V . Divide \mathcal{F} into $n+1$ sets $\mathcal{F}_0, \mathcal{F}_1, \dots, \mathcal{F}_n$, where H is in \mathcal{F}_i if and only if $\dim_{\mathbb{C}}(V^H) = i$. If N is an $\mathcal{O}_{\mathcal{F}}$ -module, then we define

$$\Sigma^V N = \bigoplus_{i=0}^n \Sigma^{2i} e_{\mathcal{F}_i} N \quad \Sigma^{-V} N = \bigoplus_{i=0}^n \Sigma^{-2i} e_{\mathcal{F}_i} N.$$

For an admissible representation V , and some $\mathcal{O}_{\mathcal{F}}$ -module N , we have a map $c^V : N \rightarrow \Sigma^{-V} N$. We define this differently on each of the subdivisions of \mathcal{F} , on part \mathcal{F}_i , we use $c^i : e_{\mathcal{F}_i} N \rightarrow \Sigma^{-2i} e_{\mathcal{F}_i} N$.

We note that c^V is not an element of $\mathcal{O}_{\mathcal{F}}$, because it does not have the same degree in each factor $\mathbb{Q}[c_H]$. For the sake of expediency, we shall often ignore this and pretend that c^V is in $\mathcal{O}_{\mathcal{F}}$. This will not cause any technical difficulties and will simplify the exposition.

Definition 3.1.3 For some function $v : \mathcal{F} \rightarrow \mathbb{Z}_{\geq 0}$ define $c^v \in \mathcal{O}_{\mathcal{F}}$ to be that element with $e_H c^v = c_H^{v(H)}$. Let \mathcal{E} be the set

$$\{c^\nu \mid \nu : \mathcal{F} \rightarrow \mathbb{Z}_{\geq 0} \text{ with finite support}\}.$$

We call this the set of **Euler classes**.

Note that the set of Euler classes is multiplicatively closed.

Example 3.1.4 The standard example of an element of \mathcal{E} comes from the dimension function of a complex representation V of \mathbb{T} with $V^{\mathbb{T}} = 0$. This function sends $H \in \mathcal{F}$ to the dimension of V^H over \mathbb{C} . We call this element c^V . Note that $c^V c^W = c^{V \oplus W}$.

For more details see [Gre99, Section 4.6]. It is important to note that every element of \mathcal{E} can be made from elements of form c^V , for V an admissible representation.

Definition 3.1.5 For N an $\mathcal{O}_{\mathcal{F}}$ -module, we define $\mathcal{E}^{-1} N$ as the colimit of terms $\Sigma^{-V} N$ as V runs over the admissible representations V with morphisms $V \rightarrow V \oplus W$ corresponding to $c^W : \Sigma^{-V} N \rightarrow \Sigma^{-(V \oplus W)} N$.

$$\mathcal{E}^{-1} N = \operatorname{colim}_{V^{\mathbb{T}}=0} \Sigma^{-V} N$$

We note that this means that $c^V : \mathcal{E}^{-1} N \rightarrow \Sigma^{-V} N$ is an isomorphism. Hence, with the viewpoint that the terms c^V are elements of $\mathcal{O}_{\mathcal{F}}$, we have inverted the set \mathcal{E} . In particular, we see that $\mathcal{E}^{-1} \mathcal{O}_{\mathcal{F}}$ is a ring. To illustrate its structure, we see that as a vector space, $(\mathcal{E}^{-1} \mathcal{O}_{\mathcal{F}})_{2n}$ is $\prod_{H \in \mathcal{F}} \mathbb{Q}$ for $n \leq 0$ and is $\bigoplus_{H \in \mathcal{F}} \mathbb{Q}$ for $n > 0$.

Definition 3.1.6 We define the category $\mathcal{A} = \mathcal{A}(\mathbb{T})$ to have object-class the collection of maps $\beta: N \rightarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes V$ of $\mathcal{O}_{\mathcal{F}}$ -modules, with V a graded rational vector space, such that $\mathcal{E}^{-1}\beta$ is an isomorphism. The $\mathcal{O}_{\mathcal{F}}$ -module N is called the **nub** and V is called the **vertex**. A map (θ, ϕ) in \mathcal{A} is a commutative square

$$\begin{array}{ccc} N & \xrightarrow{\beta} & \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes V \\ \theta \downarrow & & \downarrow \text{Id} \otimes \phi \\ N' & \xrightarrow{\beta'} & \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes V' \end{array}$$

where θ is a map of $\mathcal{O}_{\mathcal{F}}$ -modules and ϕ is a map of graded rational vector spaces.

The relation between this category and rational \mathbb{T} -equivariant spectra is given by the following pair of theorems from [Gre99].

Theorem 3.1.7 (Greenlees) *The homotopy category of rational \mathbb{T} -equivariant spectra is equivalent to the category \mathcal{A} .*

For a rational \mathbb{T} -equivariant spectrum X , $\pi_*^{\mathcal{A}}(X)$ is the following object of \mathcal{A} . See definition 4.2.1 for details of the spectra $D_S E\mathcal{F}_+$ and $\tilde{E}\mathcal{F}$.

$$\pi_*^{\mathcal{A}}(X) = (\pi_*^{\mathbb{T}}(X \wedge D_S E\mathcal{F}_+) \longrightarrow \pi_*^{\mathbb{T}}(X \wedge D_S E\mathcal{F}_+ \wedge \tilde{E}\mathcal{F}) \cong \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes \pi_*(\Phi^{\mathbb{T}} X))$$

There is also an Adams short exact sequence which explains how to calculate maps in the homotopy category of rational \mathbb{T} -equivariant spectra.

Theorem 3.1.8 (Greenlees) *Let X and Y be \mathbb{T} -equivariant spectra, then the sequence below is exact.*

$$0 \rightarrow \text{Ext}_{\mathcal{A}}(\pi_*^{\mathcal{A}}(\Sigma X), \pi_*^{\mathcal{A}}(Y)) \rightarrow [X, Y]_*^{\mathbb{T}} \otimes \mathbb{Q} \rightarrow \text{Hom}_{\mathcal{A}}(\pi_*^{\mathcal{A}}(X), \pi_*^{\mathcal{A}}(Y)) \rightarrow 0$$

In [Gre99] a model structure is also given for the category of objects in \mathcal{A} that have a differential. We define what it means to have a differential and then introduce the model structure.

If we think of $\mathcal{O}_{\mathcal{F}}$ as an object of $\text{Ch}_{\mathbb{Q}}$ with trivial differential, then we can consider the category of $\mathcal{O}_{\mathcal{F}}$ -modules in $\text{Ch}_{\mathbb{Q}}$. Such an object N is an $\mathcal{O}_{\mathcal{F}}$ -module in graded vector spaces along with maps $\partial_n: N_n \rightarrow N_{n-1}$. These maps satisfy the relations below.

$$\partial_{n-1} \circ \partial_n = 0 \quad c\partial_n = \partial_{n-2}c$$

Definition 3.1.9 *The category $\partial\mathcal{A}$ has object-class the collection of maps $\beta: N \rightarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes V$ of $\mathcal{O}_{\mathcal{F}}$ -modules in $\text{Ch}_{\mathbb{Q}}$, where N is a rational chain complex with an action of $\mathcal{O}_{\mathcal{F}}$, V is a rational chain complex and $\mathcal{E}^{-1}\beta$ is an isomorphism. A map (θ, ϕ) is then a commutative square as before, such that θ is a map in the category of $\mathcal{O}_{\mathcal{F}}$ -modules in $\text{Ch}_{\mathbb{Q}}$ and ϕ is a map of $\text{Ch}_{\mathbb{Q}}$.*

For A and B in $\partial\mathcal{A}$, we define $\mathcal{A}(A, B)_*$ to be the graded set of maps from the underlying object of A in \mathcal{A} to the underlying object of B in \mathcal{A} . We equip this graded \mathbb{Q} -module with the differential induced by the convention $df_n = d_B f_n + (-1)^{n+1} f_n d_A$. By considering an object of \mathcal{A} as an object of $\partial\mathcal{A}$ with no differential, we can extend the definition of $\mathcal{A}(A, B)_*$ to allow for the case where A is in \mathcal{A} .

Proposition 3.1.10 *The category $\partial\mathcal{A}$ has a model structure where the cofibrations are the monomorphisms and the weak equivalences are the quasi-isomorphisms. This is called the **injective model structure**, we write $\partial\mathcal{A}_i$ to denote this model structure.*

As we shall see shortly, the category \mathcal{A} has a monoidal product, which induces a monoidal product on $\partial\mathcal{A}$. But the injective model structure does not make $\partial\mathcal{A}$ into a monoidal model category. The problem is torsion, and is analogous to the fact that injective model structure on $\text{Ch}\mathbb{Z}$ is not monoidal. This is a serious defect, as we are unable to effectively compare this monoidal product to the smash product of \mathbb{T} -spectra. This defect is further complicated by the lack of projective objects of \mathcal{A} . Our aim is to find a cofibrantly generated monoidal model structure on $\partial\mathcal{A}$ which is Quillen equivalent to the injective model structure. This is a necessary step in our plan to give a monoidal classification for rational \mathbb{T} -spectra and rational $O(2)$ -spectra.

We now turn to some technical tasks that will culminate in a useful result on model structures on $\partial\mathcal{A}$. Specifically, we prove that \mathcal{A} and $\partial\mathcal{A}$ are locally presentable categories, then we show that the homology isomorphisms form an accessible category. These two facts will allow us to use Smith's theorem, which appears as [Bek00, Theorem 1.7], to find model structures on $\partial\mathcal{A}$. These two set-theoretic conditions are explained in detail in [Bor94, Sections 5.2 and 5.3]. Since we only need these two terms for the technical conditions of Smith's theorem, we do not go into full detail.

By [Bek00, Proposition 3.10] we see that \mathcal{A} is locally presentable if and only if it has a set of objects G_i such that $\mathcal{A}(\oplus_i G_i, -)$ is a faithful functor from \mathcal{A} to sets. Such a set exists by [Gre99, Lemma 22.3.4] which says that there are enough wide spheres (see definition 3.3.6). To extend this to $\partial\mathcal{A}$ we take the generating set to be the collection of wide spheres tensored with $D^1 \in \text{Ch}\mathbb{Q}$.

We need to know that the set of homology isomorphisms of $\partial\mathcal{A}$ satisfies the solution set condition. We follow [Bek00, Proposition 3.13], this applies [Bek00, Proposition 1.15] to see that all one needs to show is that the homology isomorphisms are an accessible category. To prove this, we use the following facts: the homology functor $H_*: \partial\mathcal{A} \rightarrow \mathcal{A}$ commutes with filtered colimits, the isomorphisms of \mathcal{A} are accessible (they are so in any locally presentable category) and [Bek00, Proposition 1.18]. We summarise this development in the following.

Proposition 3.1.11 *Let I be a set of monomorphisms such that the maps with the right lifting property with respect to I are homology isomorphisms. Then there is a*

cofibrantly generated model structure on $\partial\mathcal{A}$ with homology isomorphisms the weak equivalences and I as the set of generating cofibrations.

3.2 Adjunctions and monoidal products

We need to know that \mathcal{A} and $\partial\mathcal{A}$ have all small limits and colimits. For that we need to relate \mathcal{A} to a larger category, which we call $\hat{\mathcal{A}}$.

We let $\hat{\mathcal{A}}$ be category of diagrams $\alpha: N \rightarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes V$ where M is an $\mathcal{O}_{\mathcal{F}}$ -module, and V is graded \mathbb{Q} -module. The map α is a map of $\mathcal{O}_{\mathcal{F}}$ -modules. A map of such diagrams is then a commutative diagram as below where f is a map of $\mathcal{O}_{\mathcal{F}}$ -modules, and g is a map of graded \mathbb{Q} -modules.

$$\begin{array}{ccc} N & \longrightarrow & \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes V \\ \downarrow f & & \downarrow \text{Id} \otimes g \\ N' & \longrightarrow & \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes V' \end{array}$$

Thus $\hat{\mathcal{A}}$ is \mathcal{A} without the restriction that the structure map of an object should be isomorphism after \mathcal{E} is inverted. There is an adjunction

$$j: \mathcal{A} \rightleftarrows \hat{\mathcal{A}}: \Gamma_h$$

where j is the inclusion, which is full and faithful. The right adjoint Γ_h takes some work to define, for full details see [Gre99, Section 20.2]. For our summary of this construction we need to introduce spheres and suspensions.

Definition 3.2.1 If $A = (\beta: N \rightarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes U)$ then we define

$$\Sigma^V A = (\beta \circ c^V: \Sigma^V N \rightarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes U)$$

We call this **suspension by the representation** V . We may also define Σ^{-V} by using c^{-V} .

We note there that since $c^V \in \mathcal{E}$, multiplication by c^V is an isomorphism on $\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}$. Hence $\Sigma^V \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \cong \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}$. Thus in the above definition we could have used $c^V \beta$ instead.

Definition 3.2.2 Define $\mathcal{O}_{\mathcal{F}}(V)$ to be the submodule of $\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}$ given by

$$\mathcal{O}_{\mathcal{F}}(V) = \{x \in \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \mid c^V x \in \mathcal{O}_{\mathcal{F}}\}$$

Finally, we set S^V to be the inclusion map below. We call this object a **representation sphere**.

$$S^V = (\mathcal{O}_{\mathcal{F}}(V) \rightarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}})$$

Lemma 3.2.3 *For V an admissible representation, multiplication by c^{-V} induces an isomorphism*

$$c^{-V} : \Sigma^V S^0 \rightarrow S^V$$

Proof This isomorphism is a more complicated version of the following simple observation. Let c have degree -2 . Define $\mathbb{Q}\langle c^{-n} \rangle$ to be the $\mathbb{Q}[c]$ submodule of $\mathbb{Q}[c, c^{-1}]$ generated by c^{-n} . So as a graded \mathbb{Q} module, $\mathbb{Q}\langle c^{-n} \rangle$ has a copy of \mathbb{Q} in every degree $2k$ for $k \leq n$. Multiplication by c^{-n} is an isomorphism

$$c^{-n} : \Sigma^{2n} \mathbb{Q}[c] \rightarrow \mathbb{Q}\langle c^{-n} \rangle. \quad \blacksquare$$

We can also define S^{-V} , the nub is the set of those $x \in \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}$ such that $c^{-V}x$ is in $\mathcal{O}_{\mathcal{F}}$. By introducing this minus sign, we get an inclusion map $S^{-V'} \rightarrow S^{-V}$ for any inclusion of admissible representations $V \rightarrow V'$.

Let $C = (P \rightarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes U)$ be an object of $\hat{\mathcal{A}}$. Now we define a graded \mathbb{Q} -module by taking a colimit over inclusions of complex representations V of \mathbb{T} such that $V^{\mathbb{T}} = 0$.

$$\mathcal{E}^{-1}P(c^0) = \operatorname{colim}_{V^{\mathbb{T}}=0} \mathcal{A}(S^{-V}, P).$$

Now we define $\Gamma_h C$ to be the left-hand vertical arrow of the following diagram

$$\begin{array}{ccc} P' & \xrightarrow{\quad} & P \\ \downarrow & & \downarrow \\ \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes \mathcal{E}^{-1}P(c^0) & \longrightarrow & \mathcal{E}^{-1}P \end{array}$$

Our first use of Γ_h is to define limits in \mathcal{A} . It is clear that the adjunction (j, Γ_h) passes to categories with differentials, as does the following definition.

Definition 3.2.4 *Let I be some small category and let $\{N_i \rightarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes V_i\}$ be the objects of some I -shaped diagram in \mathcal{A} . The colimit over I is*

$$\operatorname{colim}_i N_i \rightarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes (\operatorname{colim}_i V_i).$$

The limit is formed by first applying the functor j , taking limits in $\hat{\mathcal{A}}$ (which are constructed termwise) and then applying Γ_h . A slightly more direct construction of limits is to take some diagram $(N_i \rightarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes V_i)$, and construct the following pullback square.

$$\begin{array}{ccc} M & \xrightarrow{\quad} & \lim(N_i) \\ f \downarrow & & \downarrow \\ \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes \lim(V_i) & \longrightarrow & \lim(\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes V_i) \end{array}$$

The limit of the I -shaped diagram, $\{N_i \rightarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes V_i\}$, is $\Gamma_h f$.

Now we turn to the monoidal product of \mathcal{A} and $\partial\mathcal{A}$.

Definition 3.2.5 *Given two elements of $\partial\mathcal{A}$, $\beta: N \rightarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes V$ and $\Gamma_h: P \rightarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes U$, their **tensor product** is*

$$\beta \otimes \Gamma_h: N \otimes_{\mathcal{O}_{\mathcal{F}}} P \rightarrow (\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes V) \otimes_{\mathcal{O}_{\mathcal{F}}} (\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes U) \cong \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes (V \otimes_{\mathbb{Q}} U)$$

The unit of this monoidal product is the object $S^0 = i: \mathcal{O}_{\mathcal{F}} \rightarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes \mathbb{Q}$.

Example 3.2.6 *The tensor product of two representation spheres S^V and S^U is the representation sphere $S^{V \oplus U}$.*

This monoidal product has homotopical meaning, as we can see from the short exact sequence of [Gre99]

$$0 \rightarrow \pi_*^{\mathcal{A}}(X) \otimes \pi_*^{\mathcal{A}}(Y) \rightarrow \pi_*^{\mathcal{A}}(X \wedge Y) \rightarrow \Sigma \text{Tor}(\pi_*^{\mathcal{A}}(X), \pi_*^{\mathcal{A}}(Y)) \rightarrow 0.$$

This monoidal structure is closed, that is, there is an internal function object describing the $\partial\mathcal{A}$ -object of maps between two objects. This functor is more complicated than the tensor product and requires use of the torsion functor Γ_h .

Definition 3.2.7 *Consider two elements of $\partial\mathcal{A}$, $A = \beta: N \rightarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes V$ and $B = \Gamma_h: P \rightarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes U$. The **function object** $F(A, B)$ is the map $\Gamma_h \delta$, where δ is defined by the pullback square below.*

$$\begin{array}{ccc} Q & \xrightarrow{\delta} & \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes \text{Hom}_{\mathbb{Q}}(V, U) \\ \downarrow & & \downarrow \\ & \text{Hom}_{\mathcal{O}_{\mathcal{F}}}(\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes V, \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes U) & \\ \downarrow & & \downarrow \\ \text{Hom}_{\mathcal{O}_{\mathcal{F}}}(N, P) & \longrightarrow & \text{Hom}_{\mathcal{O}_{\mathcal{F}}}(N, \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes U) \end{array}$$

The monoidal product and function object are related by a natural isomorphism. Let A , B and C denote objects of $\partial\mathcal{A}$, then

$$\partial\mathcal{A}(A \otimes B, C) \cong \partial\mathcal{A}(A, F(B, C)).$$

There is an adjoint pair relating $\text{Ch}_{\mathbb{Q}}$ and $\partial\mathcal{A}$ which makes $\partial\mathcal{A}$ into a closed $\text{Ch}_{\mathbb{Q}}$ -module, in the sense of [Hov99, Section 4.1]. For $K \in \text{Ch}_{\mathbb{Q}}$ we define $LK \in \partial\mathcal{A}$ as

$$i \otimes \text{Id}_K: \mathcal{O}_{\mathcal{F}} \otimes K \rightarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes K.$$

For A and B in \mathcal{A} , we define $\mathcal{A}(A, B)_*$ to be the graded set of maps of \mathcal{A} (ignoring the differential). We then equip this chain complex with the differential induced by the

convention $df_n = d_B f_n + (-1)^{n+1} f_n d_A$. The right adjoint R to the functor L sends an object A of \mathcal{A} to the $\text{Ch}_{\mathbb{Q}}$ object $\mathcal{A}(S^0, A)_*$.

For $K \in \text{Ch}_{\mathbb{Q}}$ and $A = (\beta: N \rightarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes V)$ in $\partial\mathcal{A}$, their tensor product is $A \otimes K := A \otimes LK$. hence it has form

$$\beta \otimes \text{Id}_K: N \otimes_{\mathbb{Q}} K \rightarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes (V \otimes_{\mathbb{Q}} K)$$

There is a cotensor product, A^K , defined in terms of the the internal function object of $\text{Ch}_{\mathbb{Q}}$: $\text{Hom}_{\mathbb{Q}}(K, RA)$. The $\text{Ch}_{\mathbb{Q}}$ -object of maps between two objects A and B of $\partial\mathcal{A}$ is $RF(A, B)$. This enrichment, tensor and cotensor are related by the natural isomorphisms below.

$$\partial\mathcal{A}(A, B^K) \cong \partial\mathcal{A}(A \otimes K, B) \cong \text{Ch}_{\mathbb{Q}}(K, RF(A, B))$$

Definition 3.2.8 *The category of **torsion** $\mathcal{O}_{\mathcal{F}}$ -modules is the full subcategory of $\mathcal{O}_{\mathcal{F}}$ -modules consisting of those N such that $\mathcal{E}^{-1}N = 0$. We write $\text{tors-}\mathcal{O}_{\mathcal{F}}\text{-mod}$ for this category. When we want to consider the category of such objects equipped with differentials we write $\text{tors-}\partial\mathcal{O}_{\mathcal{F}}\text{-mod}$.*

Now we offer three more adjoint pairs. We define them in terms of \mathcal{A} , but the pairs clearly extend to the categories with differentials.

There is an adjoint pair relating the category of torsion modules to \mathcal{A} . The right adjoint e takes a torsion module T to the object $T \rightarrow 0$. The left adjoint of this functor takes $\beta: N \rightarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes V$ to the kernel of β .

There is another adjoint pair relating graded rational vector spaces to \mathcal{A} . The right adjoint f takes V to the object $\text{Id}: \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes V \rightarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes V$. The left adjoint to this functor sends $\beta: N \rightarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes V$ to V .

The third adjoint pair, relates \mathcal{A} to the whole category of $\mathcal{O}_{\mathcal{F}}$ -modules. Given any $\mathcal{O}_{\mathcal{F}}$ -module N there is an object $N \rightarrow 0$ of $\hat{\mathcal{A}}$. We must apply the functor Γ_h to this to obtain an object of \mathcal{A} . We call this composite functor g and note that it is right adjoint to the functor which sends an object of \mathcal{A} to its nub. The object $\Gamma_h(N \rightarrow 0)$ is relatively easy to describe, since $\mathcal{E}^{-1}(N)(c^0) = \mathcal{E}^{-1}N$. Hence $\Gamma_h(N \rightarrow 0)$ is defined as the left hand vertical in the pullback diagram below.

$$\begin{array}{ccc} P & \xrightarrow{\quad} & N \\ \downarrow & & \downarrow \\ \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes \mathcal{E}^{-1}N & \longrightarrow & \mathcal{E}^{-1}N \end{array}$$

Lemma 3.2.9 *The functor g from the category of $\mathcal{O}_{\mathcal{F}}$ -modules is exact.*

Proof The functor sending an $\mathcal{O}_{\mathcal{F}}$ -module N to $N \rightarrow 0$ in $\hat{\mathcal{A}}$ is clearly exact. By [Gre99, Proposition 20.3.4], we see that the right derived functors of g are all zero. Hence we see that g is exact. ■

3.3 Dualisable objects of $\mathcal{A}(\mathbb{T})$

We introduce a useful class of objects with well-behaved monoidal properties. We will need to understand this class to obtain a monoidal model structure on $\partial\mathcal{A}$.

Definition 3.3.1 *A object A of \mathcal{A} is said to be **dualisable** if for any $B \in \mathcal{A}$ the canonical map $F(A, S^0) \otimes B \rightarrow F(A, B)$ is an isomorphism. The **dual** of an object B is the object $DB := F(B, S^0)$.*

Clearly, S^0 is dualisable. We now give some nice properties of dualisable objects.

Lemma 3.3.2 *Let A be dualisable. Then DA is dualisable, $D(DA) \cong A$ and A is flat. For any B and C in \mathcal{A} $F(B, A \otimes C) \cong F(B \otimes DA, C)$.*

Proof Most of these statements are proven in [LMSM86, Section III.1]. To see that dualisable implies flat, we must prove that $A \otimes -$ is an exact functor. It is always right exact and it is isomorphic to $F(DA, -)$, which is always left exact. ■

The ring $\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}$ is the colimit over admissible representations V of $\Sigma^{-V}\mathcal{O}_{\mathcal{F}}$. Indeed, for any $\mathcal{O}_{\mathcal{F}}$ -module M , $\mathcal{E}^{-1}M$ is the colimit over admissible V of $\Sigma^{-V}M$. Hence, if N is finitely presented, then

$$\mathrm{Hom}_{\mathcal{O}_{\mathcal{F}}}(N, \mathcal{E}^{-1}M) \cong \mathrm{colim}_V \Sigma^{-V} \mathrm{Hom}_{\mathcal{O}_{\mathcal{F}}}(N, M) \cong \mathcal{E}^{-1} \mathrm{Hom}_{\mathcal{O}_{\mathcal{F}}}(N, M)$$

Recall that a finitely generated projective module is finitely presented. These two facts allow us to prove the following analogue of [Hov04, Propositions 1.3.2, 1.3.3 and 1.3.4].

Proposition 3.3.3 *Let $A = (\beta: N \rightarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes V)$ and $B = (\Gamma_h: M \rightarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes U)$ be objects of \mathcal{A} and assume that the nub of A is finitely presented and has no \mathcal{E} -torsion. Then $F(A, B)$ is canonically isomorphic to*

$$\mathrm{Hom}_{\mathcal{O}_{\mathcal{F}}}(N, M) \longrightarrow \mathcal{E}^{-1} \mathrm{Hom}_{\mathcal{O}_{\mathcal{F}}}(N, M) \cong \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes \mathrm{Hom}_{\mathbb{Q}}(V, U)$$

Proof We note that since N is finitely presented, V must be finite dimensional. Hence the diagonal map

$$\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes \mathrm{Hom}_{\mathbb{Q}}(V, U) \longrightarrow \mathrm{Hom}_{\mathcal{O}_{\mathcal{F}}}(\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes V, \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes U)$$

is an isomorphism. The target of this map is isomorphic to the domain of the map below, which is induced by $\alpha: N \rightarrow \mathcal{E}^{-1}N$.

$$\alpha^*: \mathrm{Hom}_{\mathcal{O}_{\mathcal{F}}}(\mathcal{E}^{-1}N, \mathcal{E}^{-1}M) \longrightarrow \mathrm{Hom}_{\mathcal{O}_{\mathcal{F}}}(N, \mathcal{E}^{-1}M)$$

Since N has no \mathcal{E} torsion, the above map is in fact an isomorphism. The module N is also finitely presented, so we see that the codomain of the above is isomorphic $\mathcal{E}^{-1} \mathrm{Hom}_{\mathcal{O}_{\mathcal{F}}}(N, M)$. Hence we have a canonical isomorphism

$$\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes \mathrm{Hom}_{\mathbb{Q}}(V, U) \longrightarrow \mathcal{E}^{-1} \mathrm{Hom}_{\mathcal{O}_{\mathcal{F}}}(N, M).$$

Recall that the definition of $F(A, B)$ is given in terms of a pullback over the diagonal map and the map α^* . It follows that $F(A, B)$ is given by applying the torsion functor to the map

$$\mathrm{Hom}_{\mathcal{O}_{\mathcal{F}}}(N, M) \longrightarrow \mathrm{Hom}_{\mathcal{O}_{\mathcal{F}}}(N, \mathcal{E}^{-1}M) \longrightarrow \mathcal{E}^{-1}\mathrm{Hom}_{\mathcal{O}_{\mathcal{F}}}(N, M)$$

but the codomain of this is canonically isomorphic to $\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes \mathrm{Hom}_{\mathbb{Q}}(V, U)$. Hence the torsion functor has no effect, so the result is proven. \blacksquare

Proposition 3.3.4 *If $F(A, -)$ preserves filtered colimits in \mathcal{A} , then A is finitely presented. If $A \in \mathcal{A}$ is finitely presented in \mathcal{A} , then its nub is finitely presented in the category of $\mathcal{O}_{\mathcal{F}}$ -modules. Conversely, if the nub of A is finitely presented in the category of $\mathcal{O}_{\mathcal{F}}$ -modules and has no \mathcal{E} -torsion, then A is finitely presented in \mathcal{A} .*

Proof The first statement is routine, take some filtered colimit diagram B_i in \mathcal{A} . Then we have isomorphisms as below.

$$\begin{aligned} \mathrm{colim}_i \mathcal{A}(A, B_i) &\cong \mathrm{colim}_i \mathcal{A}(S^0, F(A, B_i)) \cong \mathcal{A}(S^0, \mathrm{colim}_i F(A, B_i)) \\ &\cong \mathcal{A}(S^0, F(A, \mathrm{colim}_i B_i)) \cong \mathcal{A}(A, \mathrm{colim}_i B_i) \end{aligned}$$

The second statement follows from the fact that the functor g from $\mathcal{O}_{\mathcal{F}}$ -mod to \mathcal{A} (which is a right adjoint) commutes with filtered colimits.

For the converse, let $A = (\beta: N \rightarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes V)$ with N finitely presented as $\mathcal{O}_{\mathcal{F}}$ -module and \mathcal{E} -torsion free. We see immediately that V is finite dimensional. Let B_i be some filtered system of elements of \mathcal{A} , with nubs M_i and vertices U_i . We must prove that the canonical map

$$\mathrm{colim}_i F(A, B_i) \longrightarrow F(A, \mathrm{colim}_i B_i)$$

is an isomorphism in \mathcal{A} . This follows immediately from the description of $F(A, B_i)$ above, and the facts that

$$\begin{aligned} \mathrm{colim}_i \mathrm{Hom}_{\mathcal{O}_{\mathcal{F}}}(N, M_i) &\cong \mathrm{Hom}_{\mathcal{O}_{\mathcal{F}}}(N, \mathrm{colim}_i M_i) \\ \mathrm{colim}_i \mathrm{Hom}_{\mathbb{Q}}(V, U_i) &\cong \mathrm{Hom}_{\mathbb{Q}}(V, \mathrm{colim}_i U_i). \end{aligned} \quad \blacksquare$$

Proposition 3.3.5 *An object A of \mathcal{A} is dualisable if and only if its nub is finitely generated and projective as an $\mathcal{O}_{\mathcal{F}}$ -module.*

Proof Let $A = (\beta: N \rightarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes V)$ and $B = (\Gamma_h: M \rightarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes U)$. Assume that N is finitely generated and projective as an $\mathcal{O}_{\mathcal{F}}$ -module. Then N is finitely presented and has no \mathcal{E} -torsion, hence

$$F(A, B) = (\mathrm{Hom}_{\mathcal{O}_{\mathcal{F}}}(N, M) \longrightarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes \mathrm{Hom}_{\mathbb{Q}}(V, U))$$

Since N is also projective, we see that $F(A, B)$ is isomorphic to

$$\mathrm{Hom}_{\mathcal{O}_{\mathcal{F}}}(N, \mathcal{O}_{\mathcal{F}}) \otimes_{\mathcal{O}_{\mathcal{F}}} M \longrightarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes \mathrm{Hom}_{\mathbb{Q}}(V, \mathbb{Q}) \otimes U$$

which is simply $DA \otimes B$, so A is dualisable.

For the converse, assume that A is dualisable. The functor $F(A, -)$ commutes with colimits as it is isomorphic to $DA \otimes -$. Hence the nub of A is finitely presented. Furthermore A is flat, so the nub of A must have no \mathcal{E} -torsion. We must now prove that the nub of A is projective.

Let E be an exact sequence in $\mathcal{O}_{\mathcal{F}}$ -modules, then we have an exact sequence gE in \mathcal{A} . Now consider $F(A, gE)$ this is isomorphic to $DA \otimes gE$, hence these sequences are exact. We also see that $F(A, gE)$ is isomorphic to $g \operatorname{Hom}_{\mathcal{O}_{\mathcal{F}}}(N, E)$ since the left adjoint to g is strong monoidal. Now we must show that $\operatorname{Hom}_{\mathcal{O}_{\mathcal{F}}}(N, E)$ is an exact sequence. This amounts to proving that if $\alpha: X \rightarrow Y$ is a map of $\mathcal{O}_{\mathcal{F}}$ -modules, such that $g\alpha$ is a surjection, then α itself is a surjection. But this follows by looking at the pullback diagrams defining $g\alpha$ and noting that the map from the nub of gY to Y is a surjection. Thus we conclude that $\operatorname{Hom}_{\mathcal{O}_{\mathcal{F}}}(N, E)$ is exact hence N is projective. ■

Note further that the structure map of any dualisable object is injective, as the nub has no \mathcal{E} -torsion.

We need to introduce a special and useful class of dualisable objects of \mathcal{A} , which are only slightly more complicated than the algebraic representation spheres. We will use this class to satisfy the assumptions of proposition 3.1.11, our result on the existence of model structures on $\partial\mathcal{A}$.

Definition 3.3.6 *A **wide sphere** is an object $S \rightarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes U$ with U a finitely generated vector space on elements u_1, \dots, u_d . The nub S is the $\mathcal{O}_{\mathcal{F}}$ submodule of $\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes U$ generated by elements $c^{a_1} \otimes u_1, \dots, c^{a_d} \otimes u_d$ for Euler classes a_i and an element $\sum_{i=1}^d \sigma_i \otimes u_i$ of $\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes U$.*

The point of wide spheres is that they are flat and there are enough wide spheres, see [Gre99, Lemma 22.3.4], hence they can be used to define a derived monoidal product. We reproduce the proof that there are enough wide spheres.

Take another object $A = (\beta: N \rightarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes V)$. We want to show that for any $n \in N$ or any $v \in V$ there is a wide sphere and a map to A such that n or v is in the image of this map. Since $\mathcal{E}^{-1}\beta$ is an isomorphism, it suffices to only consider elements of the nub.

So consider $n \in N$ with $\beta(n) = \sum_{i=1}^d \sigma_i \otimes v_i$. For each i there is an element $p_i \in N$ with $\beta(p_i) = c^{b_i} \otimes v_i$, for b_i some Euler class. We may assume that we have made the b_i so that $\sigma_i c^{b_1+\dots+b_d}/c^{b_i} p_i \in \mathcal{O}_{\mathcal{F}}$. We can always multiply the b_i by some Euler class so that this holds. Now we must find another Euler class, we know that

$$\beta \left(\sum_{i=1}^d \sigma_i c^{b_1+\dots+b_d}/c^{b_i} p_i \right) = \sum_{i=1}^d \sigma_i c^{b_1+\dots+b_d} \otimes v_i = \beta \left(c^{b_1+\dots+b_d} p \right)$$

and since $\mathcal{E}^{-1}\beta$ is an isomorphism, there must be some Euler class b such that

$$\sum_{i=1}^d c^b \sigma_i c^{b_1+\dots+b_d}/c^{b_i} p_i = c^b c^{b_1+\dots+b_d} p.$$

Now we can define our wide sphere. Let U be the subspace of V generated by the elements v_i . Let S be the $\mathcal{O}_{\mathcal{F}}$ submodule of $\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes V$ generated by the elements $\sum_{i=1}^d \sigma_i \otimes v_i$ and $c^{b+b_1+\dots+b_d} \otimes v_i$. The structure map is the inclusion and is clearly an isomorphism after inverting \mathcal{E} .

We are ready to describe our desired map from this wide sphere to A . On the nub, it sends $\sum_{i=1}^d \sigma_i \otimes v_i$ to n and $c^{b+b_1+\dots+b_d} \otimes v_i$ to $c^{b+b_1+\dots+b_d} c^{-b_i} p_i$. On the vertex it is the inclusion. It is a useful exercise to check that this defines a map in \mathcal{A} from the wide sphere to A . The Euler classes b_i and b are needed to ensure that the non-trivial relation between $\sum_{i=1}^d \sigma_i \otimes v_i$ and the terms $c^{b+b_1+\dots+b_d} \otimes v_i$ in the nub of the wide sphere is replicated by their images in N .

Corollary 3.3.7 *Any wide sphere is dualisable. The collection of isomorphism classes of dualisable objects is a set.*

Proof The wide spheres are finitely presented, as any map out of a wide sphere is determined by a finite number of elements. The nubs are projective by [Gre99, Lemma 23.3.3], hence the wide spheres are dualisable.

The nub of any dualisable object is finitely presented, Hence there is only a set of such objects, up to isomorphism. ■

3.4 The quasi-dualisable model structure

Now we are ready to use the dualisable objects to make our desired monoidal model structure on $\mathcal{A}(\mathbb{T})$.

Choose a set of representatives for the isomorphisms classes of dualisable objects. Call this set \mathcal{P} .

Theorem 3.4.1 *There is a cofibrantly generated monoidal model structure on $\partial\mathcal{A}$, with weak equivalences the homology isomorphisms. The generating cofibrations have form $S^{n-1} \otimes P \rightarrow D^n \otimes P$ for $P \in \mathcal{P}$ and $n \in \mathbb{Z}$. We call this model structure the **quasi-dualisable model structure** and denote it $\partial\mathcal{A}_{qd}$.*

Proof We have a set of generating cofibrations I and we must show that any map $f: A \rightarrow B$ with the right lifting property with respect to I is a homology isomorphism. We see that such a map must have the property that, for any dualisable object P , the induced map of chain complexes

$$f_*: \mathcal{A}(P, X)_* \rightarrow \mathcal{A}(P, Y)_*$$

is a homology isomorphism. Let Z be the cofibre of f , then $\mathcal{A}(P, Z)_*$ is acyclic for any dualisable P . We must show that this implies that Z has no homology. It suffices to show that any cycle n in the nub of Z is also a boundary. There is a map α from

a wide sphere P to Z with this cycle in its image. By making some careful choices in its construction, we can ensure that it is a cycle in $\mathcal{A}(P, Z)_*$. Specifically, we repeat the construction detailed above, when we choose the p_i , we see that ∂p_i must be Euler torsion, so there is some Euler class r_i with $c^{r_i} p_i$ a cycle. We can then use these elements instead of the p_i in the construction of the wide sphere. Since $\mathcal{A}(P, Z)_*$ has no homology, α is a boundary, hence so is n . ■

There is also a relative projective model structure, using the same cofibrations as for the quasi-dualisable model structure, but with generating acyclic cofibrations given by $0 \rightarrow D^n P$ for $P \in \mathcal{P}$ and $n \in \mathbb{Z}$. We will make no use of this model structure. We will only consider the quasi-dualisable model structure constructed above and the injective model structure of [Gre99], which we write as $\partial \mathcal{A}_i$.

Lemma 3.4.2 *The identity functor from $\partial \mathcal{A}_{qd}$ to $\partial \mathcal{A}_i$ is the left adjoint of a Quillen equivalence.*

$$\text{Id} : \partial \mathcal{A}_{qd} \rightleftarrows \partial \mathcal{A}_i : \text{Id}$$

Lemma 3.4.3 *There is a strong symmetric monoidal Quillen pair*

$$L : \text{Ch}_{\mathbb{Q}} \rightleftarrows \partial \mathcal{A}_{qd} : R$$

where $LV = S^0 \otimes V$ and $RA = \mathcal{A}(S^0, A)_*$. Thus, $\partial \mathcal{A}_{qd}$ is a closed $\text{Ch}_{\mathbb{Q}}$ -model category.

A more general version of the above is the statement that if $i : V \rightarrow V'$ is a cofibration in $\text{Ch}_{\mathbb{Q}}$ and P is a dualisable object, then $i \otimes P$ is a cofibration of $\partial \mathcal{A}_{qd}$.

Lemma 3.4.4 *A cofibration of $\partial \mathcal{A}_{qd}$, is, on the nubs, a cofibration of $\partial \mathcal{O}_{\mathcal{F}}\text{-mod}$. If $A \in \partial \mathcal{A}_{qd}$ is cofibrant, then $A \otimes -$ preserves weak equivalences.*

Proof We claim that that if P is the nub of a dualisable object of \mathcal{A} , then it is cofibrant as an object of $\mathcal{O}_{\mathcal{F}}\text{-mod}$. Consider a lifting problem with $f : X \rightarrow Y$ an acyclic fibration of $\partial \mathcal{O}_{\mathcal{F}}\text{-mod}$. We know that $\text{Hom}_{\mathcal{O}_{\mathcal{F}}}(P, -) \cong DP \otimes -$ is an exact functor, so $\text{Hom}_{\mathcal{O}_{\mathcal{F}}}(P, f)$ is an acyclic fibration. Thus we can solve the lifting problem and our claim is true. A generating cofibration of $\partial \mathcal{A}_{qd}$ has form $S^{n-1} \otimes P \rightarrow D^n \otimes P$, hence the result follows immediately.

The second statement follows from the fact that if N is a cofibrant object of $\partial \mathcal{O}_{\mathcal{F}}\text{-mod}$, then $- \otimes N$ preserves homology isomorphisms. ■

Lemma 3.4.5 *The quasi-dualisable model structure is a proper monoidal model category that satisfies the monoid axiom.*

Proof Since the cofibrations are contained in the monomorphisms, left properness follows as usual. For right properness, note that the acyclic cofibrations of the quasi-dualisable model structure contain the maps $0 \rightarrow D^n \otimes P$ for any $P \in \mathcal{P}$. Hence it is

easily seen that the fibrations are, in particular, surjections. Right properness then follows from the facts that in $\hat{\mathcal{A}}$ pullbacks along surjections preserve quasi-isomorphisms and Γ_h preserves quasi-isomorphisms.

For the pushout product axiom, the unit is cofibrant, and the pushout of two cofibrations is again a cofibration, by the pushout product axiom for $\text{Ch}_{\mathbb{Q}}$. Now consider the pushout of an acyclic cofibration and a generating cofibration. It is routine to check that the domain and codomain of the pushout product both have trivial homology, hence the map is a weak equivalence.

For the monoid axiom, note that for any generating cofibration i and any $A \in \partial\mathcal{A}$, the map $i \otimes A$ is a monomorphism. It follows that for an acyclic cofibration j , $j \otimes A$ is a monomorphism.

By [Bar10, Corollary 2.7] we may assume that the domains of the generating acyclic cofibrations are cofibrant. Hence the cofibre, Cj , of a generating acyclic cofibration j is both cofibrant and acyclic. Now consider $Cj \otimes A$, since Cj is cofibrant, this is weakly equivalent to $Cj \otimes QA$, for QA a cofibrant replacement of A . But Cj is acyclic, so $Cj \otimes QA$ is acyclic.

Thus any map of form $j \otimes A$ is a monomorphism and a quasi-isomorphism. Such maps are closed under pushouts and transfinite compositions, so $\partial\mathcal{A}_{qd}$ satisfies the monoid axiom. \blacksquare

Lemma 3.4.6 *There is a model structure on $\partial\hat{\mathcal{A}}$ with cofibrations and weak equivalences defined objectwise. If we denote this model structure $\partial\hat{\mathcal{A}}_{ip}$, then there is a Quillen pair*

$$j : \partial\mathcal{A}_{qd} \rightleftarrows \partial\hat{\mathcal{A}}_{ip} : \Gamma_h$$

This result holds since j preserves cofibrations and homology isomorphisms.

We will later alter the model structure on $\partial\hat{\mathcal{A}}$ to turn this Quillen pair into a Quillen equivalence. We shall do so by a right Bousfield localisation in subsection 4.6.

3.5 Model structures on $\mathcal{A}(\mathcal{C})$

Now that we have a good model structure on $\partial\mathcal{A}$, we are ready to deal with the $O(2)$ case. The work of [Gre98b] tells us that the algebraic model for cyclic $O(2)$ -spectra should be a W -twisted version of \mathcal{A} . This notion of twisting is made clear in [Bar08b, Part III], but since we do not need these results in their full generality, we give a direct construction below.

Definition 3.5.1 *The category $\text{Ch}_{\mathbb{Q}}[W]$ is the category of rational chain complexes that have an action of the group of order two. This is a monoidal category with tensor product given by tensoring over \mathbb{Q} and using the diagonal W -action. The unit of this product is \mathbb{Q} in degree zero with trivial W -action.*

There is a proper, cofibrantly generated monoidal model structure on this category that satisfies the monoid axiom. The generating cofibrations are given by the maps $S^{n-1} \otimes \mathbb{Q}[W] \rightarrow D^n \otimes \mathbb{Q}[W]$ and the acyclic cofibrations are given by $0 \rightarrow D^n \otimes \mathbb{Q}[W]$ for $n \in \mathbb{Z}$. The weak equivalences are the homology isomorphisms and the fibrations are the surjections. The cofibrations are dimensionwise split injections with cofibrant cokernel. Since \mathbb{Q} is a retract of $\mathbb{Q}[W]$ we see that $S^{n-1}\mathbb{Q} \rightarrow D^n\mathbb{Q}$ is also a cofibration of $\text{Ch}_{\mathbb{Q}}[W]$. Hence the cofibrant objects do not have to be W -free.

The forgetful functor from $\text{Ch}_{\mathbb{Q}}[W]$ to $\text{Ch}_{\mathbb{Q}}$ is the right adjoint of a strong monoidal Quillen pair. The left adjoint, which we write as $W \otimes -$ sends a chain complex X to $X \oplus X$ with W acting as the swap map.

We can think of $\mathbb{Q}[c]$ as an object of this category with c of degree -2 and W acting on c by -1 . It follows that $\mathbb{Q}[c]$ is a commutative monoid in $\mathbb{Q}[W]$. Similarly $\mathcal{O}_{\mathcal{F}}$ is a commutative monoid in $\mathbb{Q}[W]$, where W acts by -1 on each c_H . We call this map $j: \mathcal{O}_{\mathcal{F}} \rightarrow \mathcal{O}_{\mathcal{F}}$.

Definition 3.5.2 *An object $A = (\beta: N \rightarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes U)$ of $\mathcal{A}(\mathcal{C})$ consists of the following data: a module N over $\mathcal{O}_{\mathcal{F}}$ in the category of graded $\mathbb{Q}[W]$ -modules, a graded rational vector space U with an action of W and a map β of $\mathcal{O}_{\mathcal{F}}$ -modules in the category of graded $\mathbb{Q}[W]$ -modules, such that $\mathcal{E}^{-1}\beta$ is an isomorphism.*

Let $B = (\Gamma_h: M \rightarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes V)$ be another object of $\mathcal{A}(\mathcal{C})$. A map in this category from $A \rightarrow B$ consists of a map $\phi: N \rightarrow M$ of $\mathcal{O}_{\mathcal{F}}$ -modules (in the category of graded $\mathbb{Q}[W]$ -modules) and a map $\theta: U \rightarrow V$ in the category of graded $\mathbb{Q}[W]$ -modules which makes the obvious square involving the structure maps commute.

The category $\partial\mathcal{A}(\mathcal{C})$ is the category of objects of $\mathcal{A}(\mathcal{C})$ equipped with differentials and morphisms which commute with the chain maps.

Note that the monoidal product of $\mathcal{A}(\mathcal{C})$ is given by taking the tensor product as in \mathcal{A} and then equipping the product with the diagonal action of W .

Lemma 3.5.3 *There is an adjoint pair relating \mathcal{A} and $\mathcal{A}(\mathcal{C})$, the left adjoint \mathbb{D} takes $\beta: N \rightarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes U$ in \mathcal{A} to the following composite.*

$$(\text{Id} \oplus j) \circ (\beta \oplus j^* \beta): N \oplus j^* N \rightarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes U \oplus j^* \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes U \rightarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes U \oplus \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes U$$

The W -action then simply swaps the two summands. The right adjoint, i^ , is the forgetful functor from $\mathcal{A}(\mathcal{C})$ to \mathcal{A} .*

The proof of the following lemma is routine, though full details are available in [Bar08b, Section 7].

Lemma 3.5.4 *There is a model structure on $\partial\mathcal{A}(\mathcal{C})$ where the weak equivalences are the homology isomorphisms and the fibrations are those maps which forget to fibrations*

of the quasi-dualisable model structure of $\partial\mathcal{A}$. This model structure is proper, cofibrantly generated, monoidal and satisfies the monoid axiom. We write it as $\partial\mathcal{A}(\mathcal{C})_{qd}$. The generating cofibrations and acyclic cofibrations are given by applying \mathbb{D} to the generating sets for the quasi-dualisable model structure on $\partial\mathcal{A}$.

Note that the ring map $j: \mathcal{O}_{\mathcal{F}} \rightarrow \mathcal{O}_{\mathcal{F}}$ induces a map $j: S^0 \rightarrow j^*S^0$. It follows that S^0 (with W -action) is a retract of $\mathbb{D}S^0$ and hence is cofibrant in $\partial\mathcal{A}(\mathcal{C})_{qd}$.

We have the following analogue of lemma 3.4.6.

Lemma 3.5.5 *There is a strong monoidal Quillen pair*

$$j: \partial\mathcal{A}(\mathcal{C})_{qd} \rightleftarrows \partial\hat{\mathcal{A}}(\mathcal{C})_{pi}: \Gamma_h.$$

The homotopy category of $\partial\mathcal{A}(\mathcal{C})_{qd}$ is equivalent to the algebraic model for cyclic spectra in [Gre99]. We use $[-, -]_*^{S_{\mathcal{C}}}$ to denote maps in the homotopy category of cyclic spectra. By keeping track of the W -action, we can think of the functor $\pi_*^{\mathcal{A}}$ as a functor from the homotopy category of cyclic spectra to the category $\mathcal{A}(\mathcal{C})$. Then [Gre99] and theorem 3.1.8 give the following.

Theorem 3.5.6 *For X and Y objects in $S_{\mathcal{C}}\text{-mod}$, there is an Adams short exact sequence as below.*

$$0 \rightarrow \text{Ext}_{\mathcal{A}(\mathcal{C})}(\pi_*^{\mathcal{A}}(\Sigma X), \pi_*^{\mathcal{A}}(Y)) \rightarrow [X, Y]_*^{S_{\mathcal{C}}} \rightarrow \text{Hom}_{\mathcal{A}(\mathcal{C})}(\pi_*^{\mathcal{A}}(X), \pi_*^{\mathcal{A}}(Y)) \rightarrow 0$$

Remark 3.5.7 *It should be possible to extend the results of this section to the case of a torus of rank r , \mathbb{T}^r . The algebraic model for \mathbb{T}^r -equivariant rational spectra is defined in [GS11]. In that paper the algebraic model is given an injective model structure where the cofibrations are the monomorphisms and the weak equivalences are the homology isomorphisms. This model structure is not monoidal. Thus if we want to study the monoidal properties of rational \mathbb{T}^r -spectra we need an analogue of the quasi-dualisable model structure.*

The key step to generalising this section to \mathbb{T}^r is showing that one has the analogue of wide spheres. These should be dualisable objects which form a set of categorical generators. Given this, the analogue of proposition 3.1.11 would follow by the same argument. Then one would need to classify the dualisable objects and show that the collection of isomorphism classes of such objects forms a set, see corollary 3.3.7. Then we can apply the arguments of this section to obtain the quasi-dualisable model structure and see that it is monoidal.

4 Cyclic spectra

We will rename the equivariant orthogonal spectrum $\mathbb{N}^{\#}S_{\mathcal{C}}$ to just $S_{\mathcal{C}}$. Since we only work with equivariant orthogonal spectra in this section, no confusion can occur. Note

that the $SO(2)$ -equivariant ring spectrum $i^*S_{\mathcal{C}}$ is the rational $SO(2)$ -equivariant sphere spectrum. Recall that we use \mathbb{T} to denote the group $SO(2)$.

We want to study the model category of cyclic $O(2)$ spectra, $S_{\mathcal{C}}\text{-mod}$. In particular, we will need to use the relation between the category of $S_{\mathcal{C}}$ -modules and rational \mathbb{T} -equivariant spectra. The idea is to lift the work of [GS11] to the category of $S_{\mathcal{C}}$ -modules. By restricting ourselves to circle group we obtain a much shorter and simpler classification. Our exposition is therefore quite different from that of Greenlees and Shipley, so we give full details.

We find it convenient to use a slightly different model structure on $S_{\mathcal{C}}\text{-mod}$, though it is Quillen equivalent to the usual one. This new model structure comes from [MM02, Theorem 6.5]. The usual model structure on $O(2)\mathcal{IS}$ has sets of generating cofibrations and acyclic cofibrations obtained by applying the suspension functors F_V to spaces of the form $O(2)/H_+ \wedge A$ for $H \leq O(2)$. If we restrict ourselves to only those with $H \leq \mathbb{T}$, we obtain a new model structure $\mathcal{C}O(2)\mathcal{IS}$. This model structure is also stable, proper, M -cellular, monoidal and satisfies the monoid axiom. Furthermore $\mathcal{C}O(2)\mathcal{IS}$ is Quillen equivalent to $S_{\mathcal{C}}\text{-mod}$ and also the model category of $S_{\mathcal{C}}$ -modules in $\mathcal{C}O(2)\mathcal{IS}$. For this section, we use that last model structure, where a weak equivalence or fibration is a map f which is a weak equivalence or fibration as a map of $\mathcal{C}O(2)\mathcal{IS}$.

The method of this section is taken from [GS11], it is the synthesis of two ideas. The first is the fact that taking fixed points gives a Quillen equivalence from $D_S E\mathbb{T}_+$ -modules in rational \mathbb{T} -equivariant spectra to $DB\mathbb{T}_+$ -modules in rational spectra. Where $DB\mathbb{T}_+$ is the dual of $B\mathbb{T}_+$ in the category of rational spectra. We shall study this Quillen equivalence in more detail later. But the key point is that it allows us to remove equivariance. The second idea is to use a Hasse square to separate the homotopical information of \mathbb{T} -equivariant spectra into pieces where we can remove equivariance without losing any information. The fundamental example of a Hasse square is below.

$$\begin{array}{ccc} \mathbb{Z} & \longrightarrow & \mathbb{Q} \\ \downarrow & & \downarrow \\ \prod_p \mathbb{Z}_p^\wedge & \longrightarrow & (\prod_p \mathbb{Z}_p^\wedge) \otimes \mathbb{Q}. \end{array}$$

This diagram can be used to decompose the data of a \mathbb{Z} -module into a rational part and p -adic parts. For \mathbb{T} -equivariant spectra, the relevant decomposition is to separate the homotopical information coming from finite subgroups from the homotopical information coming from the whole group. It turns out that to achieve this separation, we will need a diagram of model categories, as opposed to a diagram of commutative rings.

4.1 Diagrams of model categories

We will use several model categories that are built from diagrams of model categories, hence we introduce the relevant structures here. Essentially we specialising [GS11, Appendix B] to the case of the circle group, while also considering monoidal structures. Consequently we omit most of the proofs. The basic diagram that we use is the pullback diagram \mathcal{P} :

$$\bullet \longrightarrow \bullet \longleftarrow \bullet.$$

Definition 4.1.1 A \mathcal{P} -*diagram of model categories* is a pair of Quillen pairs

$$\begin{array}{ccc} L : \mathcal{A} & \rightleftarrows & \mathcal{B} : R \\ F : \mathcal{C} & \rightleftarrows & \mathcal{B} : G \end{array}$$

With L and F the left adjoints. We will usually draw this as the diagram below.

$$\mathcal{A} \begin{array}{c} \xrightarrow{L} \\ \xleftarrow{R} \end{array} \mathcal{B} \begin{array}{c} \xleftarrow{F} \\ \xrightarrow{G} \end{array} \mathcal{C}$$

Definition 4.1.2 Given a \mathcal{P} -diagram of model categories R^\bullet as above, we can define a new category, $R^\bullet\text{-mod}$. The objects of this category are pairs of morphisms, $\alpha : La \rightarrow b$ and $\gamma : Fc \rightarrow b$ in \mathcal{B} . We write such an object as $(a, \alpha, b, \gamma, c)$. A morphism in $R^\bullet\text{-mod}$ from $(a, \alpha, b, \gamma, c)$ to $(a', \alpha', b', \gamma', c')$ is a triple of maps $x : a \rightarrow a'$ in \mathcal{A} , $y : b \rightarrow b'$ in \mathcal{B} , $z : c \rightarrow c'$ in \mathcal{C} such that we have a commuting diagram in \mathcal{B}

$$\begin{array}{ccccc} La & \xrightarrow{\alpha} & b & \xleftarrow{\gamma} & Fc \\ \downarrow Lx & & \downarrow y & & \downarrow Fz \\ La' & \xrightarrow{\alpha'} & b' & \xleftarrow{\gamma'} & Fc' \end{array}$$

Note that we could also have talked of an object as a sequence $(a, \bar{\alpha}, b, \bar{\gamma}, c)$. where $\bar{\alpha} : a \rightarrow Rb$ is a map in \mathcal{A} and $\bar{\gamma} : c \rightarrow Gb$ is a map in \mathcal{C} .

We say that a map (x, y, z) in R^\bullet is an objectwise cofibration if x is a cofibration of \mathcal{A} , y is a cofibration of \mathcal{B} and z is a cofibration of \mathcal{C} . We can also define objectwise weak equivalences similarly.

Lemma 4.1.3 Consider a \mathcal{P} -diagram of model categories R^\bullet as below, with each category M -cellular and proper.

$$\mathcal{A} \begin{array}{c} \xrightarrow{L} \\ \xleftarrow{R} \end{array} \mathcal{B} \begin{array}{c} \xleftarrow{F} \\ \xrightarrow{G} \end{array} \mathcal{C}$$

Then the category of R^\bullet -modules is a M -cellular proper model category with cofibrations and weak equivalences defined objectwise. Let $I_{\mathcal{A}}$, $J_{\mathcal{A}}$ be the sets of generating cofibrations and generating acyclic cofibrations for \mathcal{A} . Similarly we have $I_{\mathcal{B}}$, $J_{\mathcal{B}}$, $I_{\mathcal{C}}$ and $J_{\mathcal{C}}$. The set of generating cofibrations for this model structure is given by the

following collection of maps: for each $i_a: a \rightarrow a'$ in $I_{\mathcal{A}}$, $i_b: b \rightarrow b'$ in $I_{\mathcal{B}}$ and each $i_c: c \rightarrow c'$ in $I_{\mathcal{C}}$ we have the maps

$$\begin{aligned} (i_a, \text{Id}, \text{Id}) &: (a, Li_a, La', *, *) \rightarrow (a', \text{Id}, La', *, *) \\ (\text{Id}, i_b, \text{Id}) &: (*, *, b, *, *) \rightarrow (*, *, b', *, *) \\ (\text{Id}, \text{Id}, i_c) &: (*, *, Fc', Fi_c, c) \rightarrow (*, *, Fc', \text{Id}, c'). \end{aligned}$$

The set of generating acyclic cofibrations is given by the following collection of maps: for each $j_a: a \rightarrow a'$ in $J_{\mathcal{A}}$, $j_b: b \rightarrow b'$ in $J_{\mathcal{B}}$ and each $j_c: c \rightarrow c'$ in $J_{\mathcal{C}}$ we have the maps

$$\begin{aligned} (j_a, \text{Id}, \text{Id}) &: (a, Lj_a, La', *, *) \rightarrow (a', \text{Id}, La', *, *) \\ (\text{Id}, j_b, \text{Id}) &: (*, *, b, *, *) \rightarrow (*, *, b', *, *) \\ (\text{Id}, \text{Id}, j_c) &: (*, *, Fc', Fj_c, c) \rightarrow (*, *, Fc', \text{Id}, c'). \end{aligned}$$

We denote this model structure $R^\bullet\text{-mod}_{pi}$ in order to mesh with the notation of [GS11].

We can also think of maps of \mathcal{P} -diagrams of model categories. Let R^\bullet and S^\bullet be two such diagrams, where R^\bullet is as above and S^\bullet is given below.

$$\mathcal{A}' \begin{array}{c} \xrightarrow{L'} \\ \xleftarrow{R'} \end{array} \mathcal{B}' \begin{array}{c} \xleftarrow{F'} \\ \xrightarrow{G'} \end{array} \mathcal{C}'$$

Now we assume that we have Quillen equivalences as below, such that P_2L is naturally isomorphic to $L'P_1$ and P_2F is naturally isomorphic to $F'P_3$.

$$\begin{aligned} P_1: \mathcal{A} &\rightleftarrows \mathcal{A}': Q_1 \\ P_2: \mathcal{B} &\rightleftarrows \mathcal{B}': Q_2 \\ P_3: \mathcal{C} &\rightleftarrows \mathcal{C}': Q_3 \end{aligned}$$

We then obtain a Quillen pair (P, Q) between $R^\bullet\text{-mod}_{pi}$ and $S^\bullet\text{-mod}_{pi}$. For example, the left adjoint P takes the object $(a, \alpha, b, \gamma, c)$ to $(P_1a, P_2\alpha, P_2b, P_2\gamma, P_3c)$. The commutativity assumptions ensure that this is an object of $S^\bullet\text{-mod}_{pi}$.

Lemma 4.1.4 *The adjunction (P, Q) between $R^\bullet\text{-mod}_{pi}$ and $S^\bullet\text{-mod}_{pi}$ is a Quillen equivalence.*

Now we can turn to monoidal considerations. There is an obvious monoidal product for $R^\bullet\text{-mod}$, provided each of \mathcal{A} , \mathcal{B} and \mathcal{C} is monoidal and left adjoints L and F are strong monoidal.

$$(a, \alpha, b, \gamma, c) \wedge (a', \alpha', b', \gamma', c') := (a \wedge a', \alpha \wedge \alpha', b \wedge b', \gamma \wedge \gamma', c \wedge c')$$

Let $S_{\mathcal{A}}$ be the unit of \mathcal{A} , $S_{\mathcal{B}}$ be the unit of \mathcal{B} and let $S_{\mathcal{C}}$ be the unit of \mathcal{C} . Since L is monoidal, we have maps $\eta_{\mathcal{A}}: LS_{\mathcal{A}} \rightarrow S_{\mathcal{B}}$ and $\eta_{\mathcal{C}}: FS_{\mathcal{C}} \rightarrow S_{\mathcal{B}}$. We then can see that the unit of the monoidal product on $R^\bullet\text{-mod}$ is $(S_{\mathcal{A}}, \eta_{\mathcal{A}}, S_{\mathcal{B}}, \eta_{\mathcal{C}}, S_{\mathcal{C}})$.

There is also an internal function object, which is more complicated to define than the monoidal product. We let $F_{\mathcal{A}}(a, a')$ denote the internal function object of \mathcal{A} , $F_{\mathcal{B}}(b, b')$ denote the internal function object of \mathcal{B} and $F_{\mathcal{C}}(c, c')$ denote the internal function object of \mathcal{C} . Then we can construct pullback diagrams as below.

$$\begin{array}{ccc}
P & \xrightarrow{\theta} & RF_{\mathcal{B}}(b, b') \\
\downarrow & & \downarrow \\
& & F_{\mathcal{B}}(Rb, Rb') \\
& & \downarrow (\alpha)^* \\
F_{\mathcal{A}}(a, a') & \xrightarrow{(\alpha')^*} & F_{\mathcal{B}}(a, Rb')
\end{array}
\quad
\begin{array}{ccc}
GF_{\mathcal{B}}(b, b') & \xleftarrow{\phi} & Q \\
\downarrow & & \downarrow \\
& & F_{\mathcal{B}}(Gb, Gb') \\
& & \downarrow (\gamma)^* \\
F_{\mathcal{A}}(c, Gb') & \xleftarrow{(\gamma')^*} & F_{\mathcal{B}}(c, c')
\end{array}$$

The internal function object is then defined as

$$F_{R^{\bullet}}((a, \alpha, b, \gamma, c), (a', \alpha', b', \gamma', c')) = (P, \theta, F(b, b'), \phi, Q)$$

Lemma 4.1.5 *Consider a \mathcal{P} -diagram of model categories, R^{\bullet} , such that each vertex is a M -cellular monoidal model category. Assume further that the two adjunctions of the diagram are strong monoidal Quillen pairs. Then $R^{\bullet}\text{-mod}_{pi}$ is also a monoidal model category. If each vertex also satisfies the monoid axiom, so does $R^{\bullet}\text{-mod}_{pi}$.*

Proof Since the cofibrations and weak equivalences are defined objectwise, the pushout product and monoid axioms hold provided they do so in each model category in the diagram R^{\bullet} . ■

We can also extend our monoidal considerations to maps of diagrams. Return to the setting of a map (P, Q) of \mathcal{P} -diagrams from R^{\bullet} to S^{\bullet} as described above. If we assume that each of the adjunctions (P_1, Q_1) , (P_2, Q_2) and (P_3, Q_3) is a symmetric monoidal Quillen equivalence, then we see that (P, Q) is a symmetric monoidal Quillen equivalence.

With these formalities out of the way, we are ready to move from the model category to $S_{\mathcal{C}}$ -modules to modules over a \mathcal{P} -diagram of model categories.

4.2 S_{Top}^{\bullet} -modules

In this subsection we replace the category of $S_{\mathcal{C}}$ -modules by a Quillen equivalent diagram of model categories of modules. This has been done for the category of $i^*S_{\mathcal{C}}$ -modules in \mathbb{T} -spectra by [GS11, Proposition 4.2]. We repeat these construction for the category of $S_{\mathcal{C}}$ -modules in $O(2)$ -spectra.

Definition 4.2.1 *Let \mathcal{F} be the collection of finite cyclic subgroups of $O(2)$, then we have a universal space for this family $E\mathcal{F}$ where $E\mathcal{F}^H$ is non-equivariantly contractible*

for each finite cyclic subgroup H and $E\mathcal{F}^K = \emptyset$ for all other subgroups K . We define $\tilde{E}\mathcal{F}$ via the cofibre sequence of $O(2)$ -spaces,

$$E\mathcal{F}_+ \rightarrow S^0 \rightarrow \tilde{E}\mathcal{F}.$$

We define $D_S E\mathcal{F}_+$ as $F(E\mathcal{F}_+, S)$, this is commutative ring spectrum.

The above cofibre sequence can be expressed as a homotopy pullback square of $O(2)$ -equivariant spectra, as in [GS11, Subsection 4.D]. This is our Hasse square for \mathbb{T} -spectra.

$$\begin{array}{ccc} S & \longrightarrow & \tilde{E}\mathcal{F} \\ \downarrow & & \downarrow \\ D_S E\mathcal{F}_+ & \longrightarrow & D_S E\mathcal{F}_+ \wedge \tilde{E}\mathcal{F}. \end{array}$$

Since we are studying cyclic spectra, we need to actually take the dual of $E\mathcal{F}_+$ in the category of $S_{\mathcal{C}}$ -modules,

Definition 4.2.2 We define $DE\mathcal{F}_+$ as the fibrant replacement of $F(E\mathcal{F}_+, S_{\mathcal{C}})$, in the category of commutative $S_{\mathcal{C}}$ -algebras. We also define $S_{\mathcal{F}}$ as the localisation of $DE\mathcal{F}_+$ with respect to $DE\mathcal{F}_+ \wedge E\tilde{\mathcal{F}}$. Thus we have a map of commutative $S_{\mathcal{C}}$ -algebras $\eta: DE\mathcal{F}_+ \rightarrow S_{\mathcal{F}}$ which is a weak equivalence after smashing with $\tilde{E}\mathcal{F}$. Furthermore, $S_{\mathcal{F}}$ is $\tilde{E}\mathcal{F}$ -local. We define $\lambda: S_{\mathcal{C}} \rightarrow S_{\mathcal{F}}$ to be the unit map.

We need to justify the assertion that $S_{\mathcal{F}}$ exists. We use the arguments of [GS11, Section 3.E] to see that $DE\mathcal{F}_+ \wedge E\tilde{\mathcal{F}}$ is a $DE\mathcal{F}_+$ -cell module. Thus we can apply [EKMM97, Theorem VIII.2.2] to obtain our localisation η .

Now we can give our diagram of model categories that separates the behaviour of the finite groups from the group \mathbb{T} . In order to construct this diagram we need to see that the left Quillen functor

$$S_{\mathcal{F}} \wedge_{S_{\mathcal{C}}} -: S_{\mathcal{C}}\text{-mod} \rightarrow S_{\mathcal{F}}\text{-mod}$$

factors over $L_{E\tilde{\mathcal{F}} \wedge S_{\mathcal{C}}} S_{\mathcal{C}}\text{-mod}$. But this holds because $S_{\mathcal{F}}$ -modules are $E\tilde{\mathcal{F}}$ -local. We also note that $L_{E\tilde{\mathcal{F}} \wedge S_{\mathcal{C}}} S_{\mathcal{C}}\text{-mod}$ is a right proper model category.

Definition 4.2.3 We define S^{\bullet} to be the following diagram of model categories

$$DE\mathcal{F}_+\text{-mod} \begin{array}{c} \xrightarrow{S_{\mathcal{F}} \wedge_{DE\mathcal{F}_+} -} \\ \xleftarrow{\eta^*} \end{array} S_{\mathcal{F}}\text{-mod} \begin{array}{c} \xleftarrow{S_{\mathcal{F}} \wedge_{S_{\mathcal{C}}} -} \\ \xrightarrow{\lambda^*} \end{array} L_{E\tilde{\mathcal{F}} \wedge S_{\mathcal{C}}} S_{\mathcal{C}}\text{-mod}$$

We thus have a M -cellular, proper, stable monoidal model category $S^{\bullet}\text{-mod}_{pi}$ that satisfies the monoid axiom.

Given any $S_{\mathcal{C}}$ -module X we have an S^\bullet -module

$$S^\bullet \wedge_{S_{\mathcal{C}}} X := (DEF_+ \wedge_{S_{\mathcal{C}}} X, \text{Id}, S_{\mathcal{F}} \wedge_{S_{\mathcal{C}}} X, \text{Id}, X).$$

This functor has a right adjoint, given by taking a pullback construction. Let $A = (a, \alpha, b, \gamma, c)$ be an S^\bullet -module. Then there is a map of $S_{\mathcal{C}}$ -modules $a \rightarrow S_{\mathcal{F}} \wedge_{DEF_+} a \rightarrow b$, where the first map induced by λ and the second map is α . Similarly, there is a map of $S_{\mathcal{C}}$ -modules $c \rightarrow S_{\mathcal{F}} \wedge_{S_{\mathcal{C}}} c \rightarrow b$, where the first map induced by η and the second map is γ . Thus we have a diagram of $S_{\mathcal{C}}$ -modules $a \rightarrow b \leftarrow c$. We denote the pullback of this diagram $\text{pb } A$. We assemble this construction into the following result, the proof of which is entirely routine.

Proposition 4.2.4 *There is a strong symmetric monoidal Quillen adjunction*

$$S^\bullet \wedge_{S_{\mathcal{C}}} - : S_{\mathcal{C}}\text{-mod} \rightleftarrows S^\bullet\text{-mod}_{pi} : \text{pb}$$

Now we want to relate this to the \mathbb{T} -equivariant version of Greenlees and Shipley. Consider S^\bullet , this is a diagram of model categories of $O(2)$ -spectra. At each vertex we can apply the forgetful functor i^* to obtain model categories of \mathbb{T} -spectra: $i^*DEF_+\text{-mod}$, and $i^*S_{\mathcal{F}}\text{-mod}$ $L_{i^*E\tilde{\mathcal{F}} \wedge i^*S_{\mathcal{C}}} i^*S_{\mathcal{C}}\text{-mod}$. We can then assemble these into a diagram of model categories i^*S^\bullet . We then have a commutative square of Quillen functors as below.

$$\begin{array}{ccc} S_{\mathcal{C}}\text{-mod} & \xrightleftharpoons[\text{pb}]{S^\bullet \wedge_{S_{\mathcal{C}}} -} & S^\bullet\text{-mod}_{pi} \\ \uparrow \scriptstyle i^* & & \uparrow \scriptstyle i^* \\ O(2)_+ \wedge_{\mathbb{T}} - & & O(2)_+ \wedge_{\mathbb{T}} - \\ \downarrow \scriptstyle i^* & & \downarrow \scriptstyle i^* \\ i^*S_{\mathcal{C}}\text{-mod} & \xrightleftharpoons[\text{pb}]{i^*S^\bullet \wedge_{i^*S_{\mathcal{C}}} -} & i^*S^\bullet\text{-mod}_{pi} \end{array}$$

A map of $S_{\mathcal{C}}\text{-mod}$ is a weak equivalence if and only if i^* of the map is a weak equivalence in $i^*S_{\mathcal{C}}\text{-mod}$. The analogous statement holds for $S^\bullet\text{-mod}_{pi}$ and $i^*S^\bullet\text{-mod}_{pi}$. Furthermore, i^* commutes with the pullback functors.

To turn the horizontal adjunctions into Quillen equivalences, we must now cellularise (right Bousfield localise) both categories on the right-hand side of the above diagram. The cellularisation exists because we have a proper M-cellular model category. The generators of $S_{\mathcal{C}}\text{-mod}$ have form $S_{\mathcal{C}} \wedge F_V O(2)_+/H_+$ for H a subgroup of \mathbb{T} and V a representation of $O(2)$. Let K_{Top} be the set of images of these objects under the functor $S^\bullet \wedge_{S_{\mathcal{C}}} -$. The elements of this set will be called **cells**. We can now give the $O(2)$ -version of [GS11, proposition 4.2].

Proposition 4.2.5 *There is a Quillen equivalence*

$$S^\bullet \wedge_{S_{\mathcal{C}}} - : S_{\mathcal{C}}\text{-mod} \rightleftarrows K_{\text{Top}}\text{-cell-} S^\bullet\text{-mod}_{pi} : \text{pb}$$

Proof Since this is our first encounter with the cellularisation principle, we will give this proof a more detailed treatment than usual. We will prove that the derived unit and derived counit are weak equivalences. Since both categories are stable, the left and right adjoints derive to exact functors that preserve coproducts. Hence the category of objects X such that the derived unit (or counit) at X is an isomorphism is a localising subcategory. So, if we can show that the derived unit (and counit) is an isomorphism for the generators then it will be an isomorphism on all objects.

So consider the unit, a generator for $S_{\mathcal{C}}\text{-mod}$ has form $S_{\mathcal{C}} \wedge F_V O(2)/H_+$. These generators are cofibrant, and they are sent to $S^\bullet \wedge F_V O(2)/H_+$ by the left adjoint. Now we must consider the derived right adjoint on an object of form $S^\bullet \wedge F_V O(2)/H_+$. We see that this is weakly equivalent to taking the homotopy pullback of the diagram below.

$$\begin{array}{ccc} & S_{\mathcal{C}} \wedge F_V O(2)/H_+ \wedge E\tilde{\mathcal{F}} & \\ & \downarrow & \\ S_{\mathcal{C}} \wedge F_V O(2)/H_+ \wedge DE\mathcal{F}_+ & \longrightarrow & S_{\mathcal{C}} \wedge F_V O(2)/H_+ \wedge E\tilde{\mathcal{F}} \wedge DE\mathcal{F}_+ \end{array}$$

Since homotopy pullbacks commute with smash products we are essentially just taking the homotopy pullback of our original Hasse square

$$DE\mathcal{F}_+ \rightarrow E\tilde{\mathcal{F}} \wedge DE\mathcal{F}_+ \leftarrow E\tilde{\mathcal{F}}$$

and smashing it with $S_{\mathcal{C}} \wedge F_V O(2)/H_+$. Since that homotopy pullback is just the sphere spectrum, we see that the derived unit is a weak equivalence.

For the derived counit we can apply the same argument, since the generators for $K_{\text{Top}}\text{-cell-}S^\bullet\text{-mod}_{p_i}$ are the elements of K_{Top} . ■

Thus we can see that the purpose of cellularisation in the above Quillen pair is to restrict ourselves to those objects on which the derived counit is a weak equivalence.

4.3 $R_{\text{Top}}^\bullet\text{-modules}$

Now we are going to remove equivariance, we do so by using the adjunction of inflation and fixed points. We must examine how this functor is defined for cyclic $O(2)$ -spectra. The functor $(-)^{\mathbb{T}}$ of [MM02, Section 3], which takes a spectrum indexed on a complete $O(2)$ -universe \mathcal{U} to the \mathbb{T} -trivial universe $\mathcal{U}^{\mathbb{T}}$, then applies the space-level fixed point functor levelwise.

Just as we have the model structure $\mathcal{C}O(2)\mathcal{IS}$, there is a free model structure on $W\mathcal{IS}$, which we write as $\mathcal{C}W\mathcal{IS}$, where the cofibrations and acyclic cofibrations are made using only spaces of form $W_+ \wedge A$. Hence cofibrant objects of this model category are

W -free. There is a Quillen pair

$$\varepsilon^* : \mathcal{C}WIS \rightleftarrows \mathcal{C}O(2)IS : (-)^\mathbb{T}.$$

Furthermore we have a diagram of Quillen functors as below, in which both the square of fixed point and forgetful functors commutes, as does the square of inflation and forgetful functors.

$$\begin{array}{ccc} \mathcal{C}WIS & \begin{array}{c} \xrightarrow{\varepsilon^* -} \\ \xleftarrow{(-)^\mathbb{T}} \end{array} & \mathcal{C}O(2)IS \\ \downarrow i^* & & \downarrow i^* \\ IS & \begin{array}{c} \xrightarrow{\varepsilon^*} \\ \xleftarrow{(-)^\mathbb{T}} \end{array} & \mathbb{T}IS \end{array}$$

Furthermore, a map f of $\mathcal{C}WIS$ is a weak equivalence if and only if i^*f is a weak equivalence of IS . Similarly, $g \in \mathcal{C}O(2)IS$ is a weak equivalence if and only if i^*g is a weak equivalence of $\mathbb{T}IS$.

Now let A be a ring spectrum in $O(2)$ -equivariant spectra. Since this adjunction is compatible with smash products, $A^\mathbb{T}$ is a ring object in W -equivariant spectra. Using [SS03a, Part (2), Theorem 3.12] we get a Quillen adjunction

$$A \wedge_{\varepsilon^* A^\mathbb{T}} \varepsilon^*(-) : A^\mathbb{T}\text{-mod} \rightleftarrows A\text{-mod} : (-)^\mathbb{T}.$$

We have two important examples of where this adjunction is a Quillen equivalence. To save space, we rename the left adjoint as \inf .

Lemma 4.3.1 *The adjunction below is a symmetric monoidal Quillen equivalence.*

$$\inf : \Phi^\mathbb{T}(DEF_+)\text{-mod} \rightleftarrows S_\mathcal{F}\text{-mod} : (-)^\mathbb{T}.$$

Proof Since $S_\mathcal{F}$ is $E\tilde{\mathcal{F}}$ -local, we see that the homotopy \mathbb{T} -fixed points functor acts as geometric fixed points. Thus we define $\Phi^\mathbb{T}(DEF_+)$ to be the \mathbb{T} -fixed points of $S_\mathcal{F}$.

The derived unit is a weak equivalence, where the generator for $\Phi^\mathbb{T}(S_\mathcal{F})\text{-mod}$ is $\Phi^\mathbb{T}(S_\mathcal{F}) \wedge W_+$.

For the derived counit, there is also only a single generator, $S_\mathcal{F} \wedge O(2)/\mathbb{T}_+$. This holds since $S_\mathcal{F} \wedge O(2)/H_+$ is trivial for any finite subgroup H of \mathbb{T} . The derived right adjoint sends this object to $\Phi^\mathbb{T}(S_\mathcal{F}) \wedge W_+$ and the result follows. \blacksquare

Lemma 4.3.2 *The adjunction below is a symmetric monoidal Quillen equivalence.*

$$\inf : DB\mathcal{F}_+\text{-mod} \rightleftarrows DEF_+\text{-mod} : (-)^\mathbb{T}.$$

Proof We define $DB\mathcal{F}_+$ to be $(DEF_+)^\mathbb{T}$. As before the derived unit is a weak equivalence on the generator, which here takes form $DB\mathcal{F}_+ \wedge W_+$. The counit is the hardest argument to make, so we transfer the argument down to the \mathbb{T} -case using i^* .

This works since weak equivalences on both sides of the adjunction are determined by applying i^* .

So we must prove that the derived counit of the adjunction below is a weak equivalence. On the left of the above we are working in non-equivariant spectra and on the right we have \mathbb{T} -equivariant spectra.

$$\inf : i^* DB\mathcal{F}_+ \text{-mod} \xrightleftharpoons{\quad} i^* DE\mathcal{F}_+ \text{-mod} : (-)^{\mathbb{T}}$$

The generators for $i^* S\mathcal{F}$ -mod have form $i^* S\mathcal{F} \wedge \mathbb{T}/H_+$. It is easy to see that the counit is a weak equivalence when $H = \mathbb{T}_+$.

By [GS11, Lemma 13.5], the class of those X such that derived counit is a weak equivalence on DX is closed under cofibre sequences and suspension by \mathbb{T} representations. This first statement is obvious, the second holds by the Thom isomorphism. Any finite \mathbb{T} -spectrum can be built from \mathbb{T}/\mathbb{T}_+ under cofibres and suspension by representations by [GS11, Lemma 13.6]. Hence the derived counit is a weak equivalence on the generators. \blacksquare

We also have the following result, which doesn't fit precisely into the above formalism, but is certainly of a similar feel. In the following the left adjoint \inf is given by $S_{\mathcal{C}} \wedge \varepsilon^*(-)$. The reason this result holds is that the derived right adjoint behaves as the geometric fixed point functor.

Lemma 4.3.3 *The adjunction below is a symmetric monoidal Quillen equivalence.*

$$\inf : \mathcal{C}WIS \xrightleftharpoons{\quad} L_{E\tilde{\mathcal{F}} \wedge S_{\mathcal{C}}} S_{\mathcal{C}} \text{-mod} : (-)^{\mathbb{T}}.$$

Proof Consider the derived unit on the generator W_+ . The left adjoint sends this object to $S_{\mathcal{C}} \wedge O(2)/\mathbb{T}_+$, now we need to understand the geometric fixed points of this object, but that is given by W_+ . It follows that the derived unit is a weak equivalence.

For the counit, we need the following fact, for any $H \in \mathcal{F}$, there is an $O(2)$ -equivariant homotopy equivalence as below.

$$E\mathcal{F}_+ \wedge O(2)/H_+ \longrightarrow O(2)/H_+$$

The generators of $L_{E\tilde{\mathcal{F}} \wedge S_{\mathcal{C}}} S_{\mathcal{C}} \text{-mod}$ have form $S_{\mathcal{C}} \wedge O(2)/H_+$ for H a subgroup of \mathbb{T} . In the localised model structure we see that $E\mathcal{F}_+$ is weakly equivalent to a point. Hence the above fact tells us that $S_{\mathcal{C}} \wedge O(2)/H_+$ is trivial in the localised model structure whenever $H \neq \mathbb{T}$. So the only generator is $S\tilde{\mathcal{F}} \wedge O(2)/\mathbb{T}_+$. The derived right adjoint sends this to W_+ and the left adjoint sends this back to $S\tilde{\mathcal{F}} \wedge O(2)/\mathbb{T}_+$. It follows that the derived counit is a weak equivalence. \blacksquare

We can extend the functor $(-)^{\mathbb{T}}$ to the level of diagrams of model categories. Note that by the definition of $DB\mathcal{F}_+$ and $\Phi^{\mathbb{T}}(DE\mathcal{F}_+)$ we have a map $\nu : DB\mathcal{F}_+ \rightarrow \Phi^{\mathbb{T}}(DE\mathcal{F}_+)$.

Definition 4.3.4 We define R^\bullet to be the diagram of model categories below.

$$D\mathcal{B}\mathcal{F}_+ - \text{mod} \begin{array}{c} \xrightarrow{\Phi^{\mathbb{T}}(DE\mathcal{F}_+) \wedge_{D\mathcal{B}\mathcal{F}_+} -} \\ \xleftarrow{\nu^*} \end{array} \Phi^{\mathbb{T}}(DE\mathcal{F}_+) - \text{mod} \begin{array}{c} \xleftarrow{\Phi^{\mathbb{T}}(DE\mathcal{F}_+) \wedge -} \\ \xrightarrow{\quad} \end{array} \mathcal{WIS}$$

Note that we have rationalised all model categories without decoration. The unmarked functor is simply the forgetful functor.

Because of the way we have constructed R^\bullet it follows that $(\inf, (-)^{\mathbb{T}})$ give a map of diagrams of model categories from R^\bullet to S^\bullet . Since each of the components is a symmetric monoidal Quillen equivalence, we immediately obtain the following.

Theorem 4.3.5 There is a symmetric monoidal Quillen equivalence.

$$R^\bullet - \text{mod}_{pi} \begin{array}{c} \xrightarrow{\inf} \\ \xleftarrow{(-)^{\mathbb{T}}} \end{array} S^\bullet - \text{mod}_{pi}$$

Of course, we want to cellularise both sides of the above equivalence, so we must specify some cells for $R^\bullet - \text{mod}_{pi}$. We define $K_{\text{Top}}^{\mathbb{T}}$ to be the set of cells given by applying the derived functor of $(-)^{\mathbb{T}}$ to K_{Top} . By [GS11, Corollary B.7] we see that the Quillen equivalence above is preserved by cellularisation.

Corollary 4.3.6 The adjunction below is a Quillen equivalence.

$$K_{\text{Top}}^{\mathbb{T}} - \text{cell} - R^\bullet - \text{mod}_{pi} \begin{array}{c} \xrightarrow{\inf} \\ \xleftarrow{(-)^{\mathbb{T}}} \end{array} K_{\text{Top}} - \text{cell} - S^\bullet - \text{mod}_{pi}$$

It is useful to have an explicit relation between our cyclic spectra version and the original \mathbb{T} -equivariant version of this result, which is [GS11, Theorem 13.2]. Recall that we have the diagram of model categories based on \mathbb{T} -spectra which we call i^*S^\bullet . We also use i^* to denote the forgetful functor from W -spectra to spectra. We can apply this to the vertices of R^\bullet to obtain a diagram of model categories based on spectra, which we call i^*R^\bullet .

We thus obtain a diagram of Quillen functors as below. We note that i^* commutes with both \inf and $(-)^{\mathbb{T}}$ up to natural isomorphism.

$$\begin{array}{ccc} R^\bullet - \text{mod}_{pi} & \begin{array}{c} \xrightarrow{\inf} \\ \xleftarrow{(-)^{\mathbb{T}}} \end{array} & S^\bullet - \text{mod}_{pi} \\ \downarrow i^* & & \downarrow i^* \\ i^*R^\bullet - \text{mod}_{pi} & \begin{array}{c} \xrightarrow{\inf} \\ \xleftarrow{(-)^{\mathbb{T}}} \end{array} & i^*S^\bullet - \text{mod}_{pi} \end{array}$$

We note that applying i^* to $K_{\text{Top}}^{\mathbb{T}}$ gives a set of cells for $i^*R^\bullet - \text{mod}_{pi}$. Equally, applying i^* gives a set of cells for $i^*S^\bullet - \text{mod}_{pi}$. These cells are compatible with the work of

[GS11] in the sense that cellularising $i^*S^\bullet\text{-mod}_{pi}$ with respect to the set i^*K_{Top} gives the model structure of [GS11, Proposition 4.2]. This holds since $i^*O(2)/H_+$ is a wedge of two copies of \mathbb{T}/H_+ . Similar statements hold for the other categories.

We have one more piece of unfinished business, which is that $K_{\text{Top}}^{\mathbb{T}}\text{-cell-}R^\bullet\text{-mod}_{pi}$ is defined in terms of spectra indexed on a complete W -universe. Note that $\mathcal{C}WIS$ on a complete W -universe is Quillen equivalent to $\mathcal{C}WIS$ on a trivial W -universe. This second category is of course just the model category of W -objects in \mathcal{IS} . Hence the change of universe functors induce a Quillen equivalence on $K_{\text{Top}}^{\mathbb{T}}\text{-cell-}R^\bullet\text{-mod}_{pi}$. We don't introduce any new notation for this change, as it is also an equivalence of categories.

4.4 R_t^\bullet -modules

We want to replace the model category $K_{\text{Top}}^{\mathbb{T}}\text{-cell-}R_{\text{Top}}^\bullet\text{-mod}_{pi}$ by a $\text{Ch}_{\mathbb{Q}}$ -model category. The results of [Shi07] and the general theory of diagrams of model categories allow us to do so. There are a large number of functors involved in this process, so we will leave the fine details to that paper.

We begin with an example of the general pattern. Note that R_{Top}^\bullet is defined by a digram of commutative rings in the category $\mathcal{C}WIS$

$$R_{\text{Top}} = \left(D B \mathcal{F}_+ \rightarrow \Phi^{\mathbb{T}}(D E \mathcal{F}_+) \leftarrow S \right)$$

where S is the sphere spectrum. Now recall the forgetful functor \mathbb{U} from orthogonal spectra with a W -action to symmetric spectra with a W -action. This is the right adjoint of symmetric monoidal Quillen equivalence. If we apply this to the above diagram we obtain a diagram of commutative ring objects in the category of symmetric spectra with a W -action

$$\mathbb{U} R_{\text{Top}} = \left(\mathbb{U} D B \mathcal{F}_+ \rightarrow \mathbb{U} \Phi^{\mathbb{T}}(D E \mathcal{F}_+) \leftarrow \mathbb{U} S \right).$$

We also obtain a diagram of model categories, $\mathbb{U} R_{\text{Top}}^\bullet\text{-mod}_{pi}$ by using the change of rings functors coming from the ring maps of R_{Top} .

$$\mathbb{U} D B \mathcal{F}_+ \text{-mod} \xrightleftharpoons[\mathbb{U} \nu^*]{\mathbb{U} \Phi^{\mathbb{T}}(D E \mathcal{F}_+) \wedge_{\mathbb{U} D B \mathcal{F}_+} -} \mathbb{U} \Phi^{\mathbb{T}}(D E \mathcal{F}_+) \text{-mod} \xrightleftharpoons[\mathbb{U} \nu^*]{\Phi^{\mathbb{T}}(D E \mathcal{F}_+) \wedge_{\mathbb{U} -}} \mathcal{C}W \text{Sp}^\Sigma$$

Furthermore since \mathbb{U} is the right adjoint of a symmetric monoidal Quillen equivalence, we obtain a symmetric monoidal Quillen equivalence between $R_{\text{Top}}^\bullet\text{-mod}_{pi}$ which is enriched over orthogonal spectra and $\mathbb{U} R_{\text{Top}}^\bullet\text{-mod}_{pi}$, which is enriched over symmetric spectra.

Since each ring is rational we can smash this diagram of commutative ring spectra $\mathbb{U} R_{\text{Top}}$ with $H\mathbb{Q}$ to obtain $H\mathbb{Q} \wedge \mathbb{U} R_{\text{Top}}$, a new diagram of commutative ring spectra.

This new diagram of rings gives a new diagram of model categories $H\mathbb{Q} \wedge UR_{\text{Top}}^\bullet$. Furthermore this is Quillen equivalent to $UR_{\text{Top}}^\bullet\text{-mod}_{pi}$. Thus we are now working with a diagram of symmetric monoidal model categories, defined by a diagram of commutative $H\mathbb{Q}$ -algebras with a W -action.

It is routine to check that Shipley's work extends to give a symmetric monoidal Quillen equivalence between the category of $H\mathbb{Q}$ -modules with a W -action and $\text{Ch}_{\mathbb{Q}}[W]$, the model category of rational chain complexes that have an action of W . So we may apply the functor Θ' of [Shi07, Theorem 1.2] to obtain a diagram of commutative rings in the category of W -objects in $\text{Ch}_{\mathbb{Q}}$, that we call R_t . We call the diagram of model categories associated to this diagram of rings R_t^\bullet . The work of Shipley also provides an isomorphism between H_*R_t and π_*R_{Top} . Moreover, there is a zig-zag of monoidal Quillen equivalences between $R_t^\bullet\text{-mod}_{pi}$ and $R_{\text{Top}}^\bullet\text{-mod}_{pi}$.

Since cellularisation is compatible with Quillen equivalences in the sense of [GS11, Corollary A.7], the work of Shipley and the above discussion gives a set of cells K_t in $R_t\text{-mod}_{pi}$ and the following result.

Proposition 4.4.1 *There is a zig-zag of Quillen equivalences between the model category $K_{\text{Top}}^\mathbb{T}\text{-cell-}R_{\text{Top}}^\bullet\text{-mod}_{pi}$ and the model category $K_t\text{-cell-}R_t^\bullet\text{-mod}_{pi}$.*

We can repeat his whole process with the diagram of ring spectra i^*R_{Top} , where i^* denotes the forgetful functor that ignores the W -action. We then obtain a zig-zag of Quillen equivalences between $i^*K_{\text{Top}}^\mathbb{T}\text{-cell-}i^*R_{\text{Top}}^\bullet\text{-mod}_{pi}$ and the model category $i^*K_t\text{-cell-}i^*R_t^\bullet\text{-mod}_{pi}$. Thus we can relate our results to those of [GS11, section 14]. In particular, the forgetful functor

$$i^* : K_t\text{-cell-}R_t^\bullet\text{-mod}_{pi} \longrightarrow i^*K_t\text{-cell-}i^*R_t^\bullet\text{-mod}_{pi}$$

preserves and detects weak equivalences and fibrations.

4.5 R_a^\bullet -modules

We want to move from the unwieldy and poorly understood diagram R_t to something simpler. To do so, we use a formality argument: we prove that the homology of R_t is so well-structured that R_t is quasi-isomorphic to its homology. Recall that by construction, we have a specified isomorphism of diagrams of rings between $H_*(R_t)$ and π_*R_{top} . We define R_a to be π_*R_{top} .

This argument is given in [GS11, Section 15] and we give a slight alteration which applies to our case of $O(2)$. The main extra point we need to know is how $W = O(2)/\mathbb{T}$ acts on $H^*(B\mathbb{T}) = \mathbb{Q}[c]$, with c of degree -2 . This is a routine calculation and we see that the non-trivial element of w is a ring map and sends c^i to $(-1)^i c$.

Lemma 4.5.1 *In each of the following c_H will be an element of degree -2 and we define $\mathcal{O}_{\mathcal{F}}$ to be the ring $\prod_{H \in \mathcal{F}} \mathbb{Q}[c_H]$.*

$$\begin{aligned}\pi_*^{\mathbb{T}}(S\mathcal{F}) &\cong \mathcal{O}_{\mathcal{F}} \\ \pi_*^{\mathbb{T}}(S\tilde{\mathcal{F}}) &\cong \mathbb{Q} \\ \pi_*^{\mathbb{T}}(S\mathcal{F} \wedge S\tilde{\mathcal{F}}) &\cong \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}\end{aligned}$$

In each case the group W acts by sending c_H to $-c_H$ and W acts trivially on \mathbb{Q} . These rings assemble into a diagram as below with each map the obvious inclusion.

$$R_a := \pi_* R_{top} = (\mathcal{O}_{\mathcal{F}} \longrightarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \longleftarrow \mathbb{Q})$$

Note that $\mathcal{O}_{\mathcal{F}}$ is a ring object in $\text{Ch}_{\mathbb{Q}}[W]$, so that $wc_H^i = (-1)^i c_H^i$.

We want to create a two-stage zig-zag of quasi-isomorphisms between R_t and R_a , we will call the intermediate term U . We draw the diagram we wish to construct below.

$$\begin{array}{ccccc} R_t^c & \longrightarrow & R_t^t & \longleftarrow & R_t^l \\ \downarrow & & \downarrow & & \downarrow \\ U^c & \longrightarrow & U^t & \longleftarrow & U^l \\ \uparrow \alpha & & \uparrow \beta & & \uparrow \gamma \\ \mathcal{O}_{\mathcal{F}} & \longrightarrow & \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} & \longleftarrow & \mathbb{Q} \end{array}$$

We start by letting $U^l = R_t^l$ and defining γ to be the unit map. For each $H \in \mathcal{F}$ there is a cycle x_H inside R_t^c which represents e_H in homology. Now consider $y_H = 1/2(x_H + wx_H)$, this is still a cycle and is W -fixed. It follows that the homology of $R_t^c[(e_H^t)^{-1}]$ is equal to e_H applied to the homology of R_t^c . Define $U^c = \prod_H R_t^c[y_H^{-1}]$, we then have a canonical map from $R_t^c \rightarrow U^c$.

For each H we pick a representative a_H in $R_t^c[y_H^{-1}]$ for the homology class of c_H . Define $b_H = 1/2(a_H - wa_H)$ to obtain a cycle on w acts as -1 . We thus have a map $\mathbb{Q}[c_H] \rightarrow R_t^c[y_H^{-1}]$ which sends c_H to b_H . Now we define α as the product over H of these maps.

We define $V^t = \prod_H R_t^t[z_H^{-1}]$ where z_H is the image of y_H in R_t^t . The map $R_t^c \rightarrow R_t^t$ induces a map $U^c \rightarrow V^t$. Now consider the set $\mathcal{E} \in \mathcal{O}_{\mathcal{F}}$, we then have the set $\alpha\mathcal{E} \in U^c$. Define $ecal_t$ to be the image of this set in V^t . We define $U^t = \mathcal{E}_t^{-1}V^t$, the map β then exists by construction. The map $U^l \rightarrow U^t$ is induced by the map $R_t^l \rightarrow R_t^t$.

A simple check shows that the maps constructed above, $f: R_a \rightarrow U$ and $g: R_t \rightarrow U$, are quasi-isomorphisms of diagrams of ring objects in $\text{Ch}_{\mathbb{Q}}[W]$. From these diagrams of rings, we can make diagrams of model categories R_a^{\bullet} and U^{\bullet} .

Proposition 4.5.2 *The adjoint pairs of extension and restriction of scalars along f and g induce symmetric monoidal Quillen equivalences between $R_t^{\bullet} \text{-mod}_{pi}$, $U^{\bullet} \text{-mod}_{pi}$*

and $R_a^\bullet\text{-mod}_{pi}$. Let K'_t be the images of the cells K_t in $U^\bullet\text{-mod}_{pi}$ and K_a the images in $R_a^\bullet\text{-mod}_{pi}$. Then we have Quillen equivalences between $K_t\text{-cell-}R_t^\bullet\text{-mod}_{pi}$, $K'_t\text{-cell-}U^\bullet\text{-mod}_{pi}$ and $K_a\text{-cell-}R_a^\bullet\text{-mod}_{pi}$.

As in previous cases, application of the forgetful functor from $\text{Ch}_{\mathbb{Q}}[W] \rightarrow \text{Ch}_{\mathbb{Q}}$ reduces us to the case of rational \mathbb{T} -spectra. We see as before that the forgetful functor

$$i^* : K_a\text{-cell-}R_a^\bullet\text{-mod}_{pi} \longrightarrow i^* K_a\text{-cell-}i^* R_a^\bullet\text{-mod}_{pi}$$

preserves and detects weak equivalences and fibrations.

4.6 Comparison with the algebraic model

We now turn to comparing $R_a^\bullet\text{-mod}_{pi}$ to the algebraic model $\partial\mathcal{A}(\mathcal{C})$. Details on this category and its model structures are in section 3. We will need some results from [GS11, Section 17] to complete this section. These results are lengthy technical results, so we simply quote them rather than give the proofs.

We first introduce an adjoint pair relating $R_a^\bullet\text{-mod}_{pi}$ and $\partial\hat{\mathcal{A}}(\mathcal{C})$. An object $\beta : M \rightarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes V$ of $\partial\hat{\mathcal{A}}(\mathcal{C})$ gives an object of $R_a^\bullet\text{-mod}_{pi}$ defined by $(M, \text{Id}, \mathcal{E}^{-1}M, \beta^{-1}, V)$. This functor, which we call k , includes $\partial\hat{\mathcal{A}}(\mathcal{C})$ into $R_a^\bullet\text{-mod}_{pi}$. This functor has a right adjoint Γ_v . Take some object of $R_a^\bullet\text{-mod}_{pi}$, $(a, \alpha, b, \gamma, c)$. Then we can draw the diagram of $\mathcal{O}_{\mathcal{F}}$ -modules

$$a \rightarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes_{\mathcal{O}_{\mathcal{F}}} a \rightarrow b \leftarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes c.$$

If we take the pullback of this in the category of $\mathcal{O}_{\mathcal{F}}$ -modules $\text{Ch}_{\mathbb{Q}}[W]$ we obtain a map $\delta : P \rightarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes c$. This map δ is an object of $\partial\hat{\mathcal{A}}(\mathcal{C})$. For more details see [Gre11, Section 7]. We call this adjoint pair (k, Γ_v) and we note that it is a strong symmetric monoidal adjunction.

We can compose this adjunction with the adjunction (j, Γ_h) which relates $\partial\hat{\mathcal{A}}(\mathcal{C})$ to $\partial\mathcal{A}(\mathcal{C})$. We let $i = k \circ j$ and $\Gamma = \Gamma_h \circ \Gamma_v$. We thus have an adjunction (i, Γ) between $\partial\mathcal{A}_i$ and $i^*R_a^\bullet\text{-mod}$ equipped with the doubly injective model structure of [GS11, Appendix B], denoted $i^*R_a^\bullet\text{-mod}_{ii}$. In this model structure the weak equivalences are defined objectwise and the cofibrations are the objectwise monomorphisms. Now we give [GS11, Proposition 16.5].

Proposition 4.6.1 *The adjunction (i, Γ) is a Quillen pair that induces a Quillen equivalence*

$$i : \partial\mathcal{A}_i \rightleftarrows i^*K_a\text{-cell-}i^*R_a^\bullet\text{-mod}_{ii} : \Gamma$$

Proof The functor i clearly preserves cofibrations and takes homology isomorphisms of $\partial\mathcal{A}_i$ to objectwise weak equivalences of $i^*R_a^\bullet\text{-mod}_{ii}$.

To see that this passes to a Quillen equivalence as stated, we need two facts. The first is that $\Gamma\sigma$ is weakly equivalent to σ for any cell $\sigma \in i^*K_a$. This holds since the homology of a cell is given by π_*^A applied to some \mathbb{T} -equivariant spectrum, see [GS11, Proposition 17.6]. The second fact is that $\partial\mathcal{A}_i$ isn't changed by cellularisation. This is the content of [GS11, Proposition 17.8], which states that the cellular equivalences of $\partial\mathcal{A}_i$ are precisely the homology isomorphisms. \blacksquare

Let K_a^{pi} denote the cofibrant replacement of the cells K_a in the model category $R_a^\bullet\text{-mod}_{pi}$. The model structure $K_a^{pi}\text{-cell-}R_a^\bullet\text{-mod}_{ii}$ is equal to $K_a\text{-cell-}R_a^\bullet\text{-mod}_{ii}$. We can also use the cells K_a^{pi} to induce a cellularisation on $\partial\mathcal{A}_{qd}$. But as with the injective model structure, this cellularisation doesn't change the model model structure. Thus we see that the pair (i, Γ) is a Quillen equivalence between $\partial\mathcal{A}_{qd}$ and $i^*K_a^{pi}\text{-cell-}i^*R_a^\bullet\text{-mod}_{pi}$. We can lift this to a Quillen equivalence between $\partial\mathcal{A}(\mathcal{C})_{pi}$ and $K_a^{qd}\text{-cell-}R_a^\bullet\text{-mod}_{pi}$ by the following standard argument. There is a commutative square of adjoint pairs as below, with the lower horizontal adjunction a Quillen equivalence.

$$\begin{array}{ccc} \partial\mathcal{A}(\mathcal{C})_{qd} & \xrightleftharpoons[\Gamma]{i} & K_a^{pi}\text{-cell-}R_a^\bullet\text{-mod}_{pi} \\ \mathbb{D} \updownarrow i^* & & \mathbb{D} \updownarrow i^* \\ \partial\mathcal{A}_{qd} & \xrightleftharpoons[\Gamma]{i} & i^*K_a^{pi}\text{-cell-}i^*R_a^\bullet\text{-mod}_{pi} \end{array}$$

The weak equivalences and fibrations for $K_a^{qd}\text{-cell-}R_a^\bullet\text{-mod}_{pi}$ and $\partial\mathcal{A}(\mathcal{C})_{qd}$ are defined in terms of the functors i^* . Thus it follows immediately that the top adjunction is also a Quillen equivalence.

We summarise this section with the following result.

Proposition 4.6.2 *The pair (i, Γ) induces a Quillen equivalence between the model categories $\partial\mathcal{A}(\mathcal{C})_{qd}$ and $K_a\text{-cell-}R_a^\bullet\text{-mod}_{pi}$.*

Remark 4.6.3 *We may ask if the results of this section can be extended to other extensions of the circle by a finite group. Let G be the extension of \mathbb{T} by a finite group F . In the rational Burnside ring of G there is an idempotent $e_{\mathcal{C}}$, which corresponds to the identity component of G , see [Bar09b, Lemma 6.6]. We can then study the model category of rational G -spectra localised at $e_{\mathcal{C}}S$. We may call this the category of cyclic G -spectra.*

A map in cyclic G -spectra is a weak equivalence if and only if when one forgets to \mathbb{T} -spectra, it is a weak equivalence. This statement is the key fact we use in this section to move from \mathbb{T} -spectra to cyclic $O(2)$ -spectra. Since it holds for general extensions G , we can apply our above results. Note that when we take \mathbb{T} -fixed points, we will obtain various model categories of rational spectra with an action of F . Similarly our algebraic model for cyclic G -spectra will be similar to $\mathcal{A}(\mathcal{C})$, where we have to account for how F acts on the cohomology of $B\mathbb{T}_+$.

Thus we can classify cyclic G –spectra in terms of an algebraic model. Furthermore this classification will respect the monoidal structures, by the results of section 5.

We can go even further and consider a finite extension of a torus of rank greater than one. The methods of this section are clearly extendible to such a setting. There are two sources of extra work, neither of which are particularly problematic. Firstly, in the case of a general torus, there are many more steps than for the circle group. Secondly, we would also need a good model structure for the algebraic model, see remark 3.5.7.

5 Symmetric monoidal equivalences

All of the Quillen pairs we have used to classify cyclic $O(2)$ –spectra have been compatible with the monoidal properties of the categories. By examining the cellularised model structures more clearly we are able to show that each of these model categories are proper, stable, M –cellular, monoidal model categories that satisfy the monoid axiom. We are thus able to conclude that our classification theorem is in terms of monoidal Quillen equivalences. It follows that we have also classified categories of ring objects and modules over ring objects.

5.1 Cellularisation of stable model categories

It is important to distinguish between K –cellular, cofibrant and K –cofibrant, the first is a condition on the homotopy type of an object, the second two are model category conditions. Of course, an object is K –cofibrant if and only if it is K –cellular and cofibrant.

Theorem 5.1.1 *Let \mathcal{C} be a proper, stable, M –cellular model category. Let K be a collection of cofibrant objects of \mathcal{C} , such that the class of K –cellular objects is closed, up to weak equivalence, by desuspension. Let I be the set of generating cofibrations, J be the set of generating acyclic cofibrations and let ΛK be a complete set of horns (see [Hir03, Definition 5.2.1]) on the set K , as below*

$$\Lambda K = \{\partial\Delta^n \otimes k \rightarrow \Delta^n \otimes k \mid n \geq 0, k \in K\}.$$

Then K –cell– \mathcal{C} is a proper, stable cellular model category with generating cofibrations given by the set $\Lambda K \cup J$ and generating acyclic cofibrations given by the set J . The weak equivalences are the K –cellular equivalences.

Proof By [Hir03, Theorem 5.1.1] there is a right proper model category K –cell– \mathcal{C} . We must show that it is stable: that the adjunction (Σ, Ω) on $\text{Ho}(K$ –cell– $\mathcal{C})$ is an equivalence. Note that if A is K –cellular, then so is ΣA by [Hir03, Lemma 5.5.2]. Let A and B be objects of \mathcal{C} , with A K –cellular, then

$$[A, B]^{K\text{--cell--}\mathcal{C}} \cong [A, B]^{\mathcal{C}} \xrightarrow{\cong} [\Sigma A, \Sigma B]^{\mathcal{C}} \cong [\Sigma A, \Sigma B]^{K\text{--cell--}\mathcal{C}}.$$

Hence Σ is full and faithful on $\text{Ho}(K\text{-cell-}\mathcal{C})$. Our assumption on closure under desuspension implies that this functor is also essentially surjective.

Now we prove that $K\text{-cell-}\mathcal{C}$ is M -cellular, see [Hir03, Definition 12.1.1]. The various smallness and compactness arguments follow from the corresponding statements for \mathcal{C} and the fact that K consists of cofibrant objects.

The K -cellular equivalences clearly satisfy the two out of three property. Now we check that the maps with the left lifting property with respect to $\Lambda K \cup J$ are precisely those maps that are fibrations of \mathcal{C} that are K -cellular equivalences.

Let $f: A \rightarrow B$ be a map with the right lifting property, then clearly f is a fibration. We must show that f is a K -cellular equivalence. Since $K\text{-cell-}\mathcal{C}$ is stable, it suffices to prove that Ff , the homotopy fibre of f , is K -cellularly trivial. We know that $Ff \rightarrow *$ is a fibration that has the right lifting property with respect to ΛK . Thus by [Hir03, Proposition 5.2.3] this map is a K -cellular equivalence. The converse holds by [Hir03, Proposition 5.2.5].

Now we must show that a J -cofibration is a $\Lambda K \cup J$ -cofibration that is a K -cellular equivalence. But this holds immediately, since every weak equivalence of \mathcal{C} is a K -cellular equivalence.

Our remaining task is to show that we have a left proper model category. Consider a pushout diagram as below, with p a K -cellular equivalence and f a cofibration of \mathcal{C} .

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow p & & \downarrow q \\ C & \xrightarrow{g} & P \end{array}$$

Let B/A be the cokernel of f , the pushout of f along $A \rightarrow *$. Since \mathcal{C} is left proper B/A is weakly equivalent to Cf , the homotopy cofibre of f and P/C is weakly equivalent to the Cg , the homotopy cofibre of g . It is a standard fact that B/A is isomorphic to P/C . Thus the induced map $Cf \rightarrow Cg$ is a weak equivalence of \mathcal{C} . Since $K\text{-cell-}\mathcal{C}$ is stable, it follows that q must also be a K -cellular equivalence. ■

The condition that the class of K -cellular objects is closed, up to weak equivalence, by desuspension, is essentially a dual version of the criteria for localised Quillen equivalences in [Hov01b, Proposition 2.3].

Proposition 5.1.2 *If, in addition, \mathcal{C} is monoidal, the unit S^0 is K -cellular and any object of the form $k \wedge k'$, for k, k' in K , is K -cellular, then $K\text{-cell-}\mathcal{C}$ is monoidal. If \mathcal{C} also satisfies the monoid axiom, then so does $K\text{-cell-}\mathcal{C}$.*

Proof If X and Y are K -cellular objects of \mathcal{C} , then $X \wedge Y$ is a homotopy colimit of objects of form $k \wedge k'$, which we have assumed to be K -cellular. Hence $X \wedge Y$ is K -cellular. We must also check that for any K -cellular X , $\widehat{c}_K S^0 \wedge X \rightarrow X$ is

a K -cellular equivalence. But this follows since $\widehat{c}_K S^0$ can be chosen to be $\widehat{c}S^0$, the cofibrant replacement of S^0 in \mathcal{C} .

Now we must check that $(\Lambda K \cup J) \square (\Lambda K \cup J)$ consists of K -cellular cofibrations. This amounts to proving that the following three collections consist of K -cellular cofibrations: $\Lambda K \square \Lambda K$, $\Lambda K \square J$ and $J \square J$. For the first, consider the cofibration

$$i = (\partial \Delta^n \otimes k \rightarrow \Delta^n \otimes k) \square (\partial \Delta^m \otimes k' \rightarrow \Delta^m \otimes k').$$

We can rewrite i , up to weak equivalence, as the following map which is a cofibration of K -cell- \mathcal{C} between K -cellular objects.

$$((\partial \Delta^n \rightarrow \Delta^n) \square (\partial \Delta^m \rightarrow \Delta^m)) \otimes (k \wedge k')$$

The domain and codomain of i are K -cellular, so by [Hir03, Proposition 3.3.16] i is a K -cellular cofibration. For the second, such a map is contained in the set of maps $I \square J$, which consist of acyclic cofibrations of \mathcal{C} and any such map is a K -cellular cofibration. For the third, such a map is an acyclic cofibration of \mathcal{C} and hence is a K -cellular cofibration. The monoid axiom holds since the set of generating acyclic cofibrations has not changed. \blacksquare

Definition 5.1.3 *Assume that K is a set of cofibrant objects in some proper, cellular, stable monoidal model category. If the class of K -cellular objects of \mathcal{C} is closed (up to weak equivalence) under desuspension we say that K is **stable**. If the unit of \mathcal{C} is K -cellular and any element of form $k \wedge k'$, for $k, k' \in K$ is K -cellular, we say that the set K is **monoidal**. We say that an object X of a stable model category \mathcal{C} is **homotopically compact** if in the homotopy category $[X, \coprod_i Y_i]^{\mathcal{C}}$ is canonically isomorphic to $\oplus_i [X, Y_i]^{\mathcal{C}}$, see [SS03b, Definition 2.1.2].*

We are now able to provide an upgraded version of the cellularisation principle, [GS11, Proposition A.6]. We now see that the Quillen equivalences created by that proposition are monoidal provided one had a monoidal context to begin with.

Proposition 5.1.4 *Consider a symmetric monoidal Quillen equivalence between a pair of proper, M -cellular, stable monoidal model categories*

$$L : \mathcal{C} \rightleftarrows \mathcal{D} : R.$$

Let K be a set of homotopically compact, cofibrant objects of \mathcal{C} . Assume further that K is stable, monoidal and LK is also stable. Then LK is a monoidal set of homotopically compact, cofibrant objects of \mathcal{D} and we have an induced symmetric monoidal Quillen equivalence

$$L : K\text{-cell-}\mathcal{C} \rightleftarrows LK\text{-cell-}\mathcal{D} : R.$$

Let H be a set of homotopically compact, cofibrant objects of \mathcal{D} . Assume further that H is stable, monoidal and $\widehat{c}R\widehat{f}H$ is also stable. Then $\widehat{c}R\widehat{f}H$ is a monoidal set

of homotopically compact, cofibrant objects of \mathcal{C} and we have an induced symmetric monoidal Quillen equivalence

$$L : \widehat{\mathcal{C}Rf}H\text{-cell-}\mathcal{C} \rightleftarrows H\text{-cell-}\mathcal{D} : R.$$

Proof Assume that we have a set K as in the first half of the proposition. Consider the set LK , this is a set of cofibrant objects of \mathcal{D} . Furthermore, $Lk \wedge Lk'$ is weakly equivalent to $L(k \wedge k')$, which by assumption is LK -cellular. The map $L(\widehat{\mathcal{C}S}_{\mathcal{C}}) \rightarrow S_{\mathcal{D}}$ is a weak equivalence of \mathcal{D} and $\widehat{\mathcal{C}S}_{\mathcal{C}}$ is K -cellular, hence $S_{\mathcal{D}}$ is LK -cellular. Hence LK is monoidal, we have assumed it to be stable, so $LK\text{-cell-}\mathcal{D}$ is a M -cellular monoidal model category.

Since (L, R) is a Quillen equivalence, we can apply [GS11, Corollary A.7] to see that (L, R) are a Quillen equivalence on the cellularised categories. Furthermore we see that every object of LK is homotopically compact in \mathcal{D} .

Now we show that the pair (L, R) induce a symmetric monoidal Quillen equivalence between the cellularised model structures. Our first step is to show that L takes K -cellular equivalences between cofibrant objects of \mathcal{C} to LK -cellular equivalences. Let $p: X \rightarrow Y$ be such a map in \mathcal{C} and consider $R\widehat{\mathcal{C}f}Lp$, This is weakly equivalent to p , which is a K -cellular equivalence. Since (L, R) is a Quillen equivalence on the cellularised model structures, it follows that Lp must be a LK -cellular equivalence.

Let X and Y be a pair of K -cellular objects of \mathcal{C} , and consider the comonoidal map $L(X \wedge Y) \rightarrow LX \wedge LY$. Since any K -cellular object of \mathcal{C} is also cofibrant, this map is a weak equivalence of \mathcal{D} . We must also prove that the counital map $L\widehat{\mathcal{C}S}_{\mathcal{C}} \rightarrow S_{\mathcal{D}}$ is an LK -cellular equivalence, for $\widehat{\mathcal{C}S}_{\mathcal{C}}$ any K -cellular replacement of the unit of \mathcal{C} . The map $L\widehat{\mathcal{C}S}_{\mathcal{C}} \rightarrow S_{\mathcal{D}}$ is a weak equivalence of \mathcal{D} and we have assumed that $\widehat{\mathcal{C}S}_{\mathcal{C}}$ is already cellular, hence the unit condition is satisfied. We have now proven that the cellularised adjunction is symmetric monoidal.

For the second case, we define $K = \widehat{\mathcal{C}Rf}H$. Since we have a Quillen equivalence, every element of this set is homotopically compact. We must show that the smash product of any two elements of K is K -cellular.

Let k and k' be elements of K then $L(k \wedge k') \rightarrow Lk \wedge Lk'$ is a weak equivalence of \mathcal{D} . We know that Lk and Lk' are H -cellular, hence so is their product. Thus we see that $L(k \wedge k')$ is H -cellular, as is $L\widehat{\mathcal{C}S}_{\mathcal{C}}(k \wedge k')$. The map $i: \widehat{\mathcal{C}S}_{\mathcal{C}}(k \wedge k') \rightarrow k \wedge k'$ is a K -cellular equivalence between cofibrant objects, hence Li is a H -cellular equivalence. The domain and codomain of Li are H -cellular, hence Li is a weak equivalence of \mathcal{D} and in turn, i must be a weak equivalence of \mathcal{C} . It follows that $k \wedge k'$ is K -cellular. We must also prove that $\widehat{\mathcal{C}S}_{\mathcal{C}}$ is K -cellular. By a lifting argument we obtain a K -cellular equivalence $\alpha: \widehat{\mathcal{C}S}_{\mathcal{C}} \rightarrow \widehat{\mathcal{C}S}_{\mathcal{C}}$, applying L to α gives a H -cellular equivalence between $L\widehat{\mathcal{C}S}_{\mathcal{C}}$ and $L\widehat{\mathcal{C}S}_{\mathcal{C}} \simeq S_{\mathcal{D}}$. The domain and codomain are H -cellular, so $L\alpha$ is a weak equivalence of \mathcal{D} , hence α is a weak equivalence of \mathcal{C} , thus we see that $\widehat{\mathcal{C}S}_{\mathcal{C}}$ is K -cellular. Hence K is monoidal and we have assumed it to be stable, so $K\text{-cell-}\mathcal{C}$

is a monoidal model category. The proof that this second adjunction is symmetric monoidal follows the same pattern as the previous case. \blacksquare

5.2 Application to the classification

We start with the adjunction

$$S^\bullet \wedge_{S_{\mathcal{C}}} - : S_{\mathcal{C}}\text{-mod} \rightleftarrows K\text{-cell-} S^\bullet\text{-mod}_{pi} : \text{pb}$$

The set of cells K is given by $S^\bullet \wedge_{S_{\mathcal{C}}} -$ applied to the generators of $S_{\mathcal{C}}\text{-mod}$. Hence the smash product of any two elements of K is K -cellular and K is closed under desuspension. The unit of $S^\bullet\text{-mod}$ is in K , hence K is stable and monoidal. Hence we have satisfied the conditions of the above results, so this adjunction is symmetric monoidal.

We then have a large number of symmetric monoidal Quillen equivalences relating $S^\bullet\text{-mod}_{pi}$ and $R_a^\bullet\text{-mod}_{pi}$. Each of these categories is stable, hence each functor that occurs in this collection derives to a functor on homotopy categories that commutes with desuspension. Since our initial set of cells K are stable, it follows that all the sets of cells that we build from K (using our various Quillen equivalences) are also stable. Our initial set of cells K is also monoidal, thus we can use proposition 5.1.4. Hence we see that the cellularisations of $S^\bullet\text{-mod}_{pi}$ and $R_a^\bullet\text{-mod}_{pi}$ are Quillen equivalent via symmetric monoidal Quillen equivalences.

It remains to compare $\partial\mathcal{A}(\mathcal{C})_{qd}$ and $K_a\text{-cell-}R_a^\bullet\text{-mod}_{pi}$. These model structures are Quillen equivalent and $K_a\text{-cell-}R_a^\bullet\text{-mod}_{pi}$ is a symmetric monoidal model category. The adjunction relating these two categories is strong symmetric monoidal and the unit of $\partial\mathcal{A}(\mathcal{C})_{qd}$ is cofibrant, hence the Quillen equivalence is symmetric monoidal.

Theorem 5.2.1 *The model category of cyclic spectra, $S_{\mathcal{C}}\text{-mod}$, is Quillen equivalent to the algebraic model $\partial\mathcal{A}(\mathcal{C})_{qd}$. Furthermore these Quillen equivalences are all symmetric monoidal, hence the homotopy categories of $S_{\mathcal{C}}\text{-mod}$ and $\partial\mathcal{A}(\mathcal{C})_{qd}$ are equivalent as symmetric monoidal categories.*

6 Dihedral spectra

In this section we find an model category, based on chain complexes of \mathbb{Q} -modules, that is monoidally Quillen equivalent to the model category of dihedral spectra $S_{\mathcal{D}}\text{-mod}$. The material we present is an updated version of the preprint [Bar08a].

6.1 The algebraic model

We take the work of [Gre98b] and consider the algebraic model for the homotopy category of dihedral spectra. We give this category a monoidal model structure.

Recall that W is used to denote the group of order two. For R a ring, let $\text{Ch}(R)$ denote the category of chain complexes of R -modules. We use $\mathbb{Q}[W]$ to denote the rational group ring of W . We will often consider rational chain complexes as objects of $\text{Ch}(\mathbb{Q}[W])$ with trivial W -action without comment or decoration.

Definition 6.1.1 *An object V of the algebraic model $\mathcal{A}(\mathcal{D})$ consists of a rational chain complex V_∞ and a collection $V_k \in \text{Ch}(\mathbb{Q}[W])$ for $k \geq 1$, with a map of chain complexes of $\mathbb{Q}[W]$ -modules $\sigma_V: V_\infty \rightarrow \text{colim}_n \prod_{k \geq n} V_k$.*

A map $f: V \rightarrow V'$ in this category consists of a map $f_\infty: V_\infty \rightarrow V'_\infty$ in $\text{Ch}(\mathbb{Q})$ and maps $f_k: V_k \rightarrow V'_k$ in $\text{Ch}(\mathbb{Q}[W])$ making the square below commute.

$$\begin{array}{ccc} V_\infty & \xrightarrow{\sigma_V} & \text{colim}_n \prod_{k \geq n} V_k \\ f_\infty \downarrow & & \downarrow \text{colim}_n \prod_{k \geq n} f_k \\ V'_\infty & \xrightarrow{\sigma_{V'}} & \text{colim}_n \prod_{k \geq n} V'_k \end{array}$$

We will also write $\text{tails}(V)$ for $\text{colim}_n \prod_{k \geq n} V_k$.

The following definition and theorem are taken from [Gre98b]. Note that for any compact Lie group G and closed subgroup H , the action of $N_G H/H$ on G/H induces an action of $N_G H/H$ on $[G/H_+, X]^G \cong \pi_*^H(X)$.

Definition 6.1.2 *For an $O(2)$ -spectrum X with rational homotopy groups, let $\underline{\pi}_*^{\mathcal{D}}(X)$ denote the following object of $\mathcal{A}(\mathcal{D})$ with trivial differential. Let $n \geq 1$ and define*

$$\underline{\pi}_*^{\mathcal{D}}(X)_k = e_{D_{2k}} \pi_*^{D_{2k}^h}(X)$$

where $e_{D_{2n}} \in A(D_{2n}^h)$ is defined in lemma 2.1.4. Recall that $f_n = e_{\mathcal{D}} - \sum_{k=1}^{n-1} e_k$ and set

$$\underline{\pi}_*^{\mathcal{D}}(X)_\infty = \text{colim}_n (f_n \pi_*^{O(2)}(X))$$

Whenever $k \geq n$, there is a map

$$f_n \pi_*^{O(2)}(X) \longrightarrow e_{D_{2k}} \pi_*^{D_{2k}^h}(X)$$

induced from the inclusion $D_{2k}^h \rightarrow O(2)$ and the application of $e_{D_{2k}}$. Thus we obtain a map

$$f_n \pi_*^{O(2)}(X) \longrightarrow \prod_{k \geq n} e_{D_{2k}} \pi_*^{D_{2k}^h}(X)$$

and taking colimits over n gives our structure map.

Since any $S_{\mathcal{D}}$ -module has rational homotopy groups this construction defines a functor

$$\underline{\pi}_*^{\mathcal{D}}: \text{Ho } S_{\mathcal{D}}\text{-mod} \rightarrow \mathcal{A}(\mathcal{D}).$$

Thus one has a map of graded \mathbb{Q} -modules $[X, Y]_*^{S_{\mathcal{D}}} \rightarrow \text{Hom}_{\mathcal{A}(\mathcal{D})}(\pi_*^{\mathcal{D}}(X), \pi_*^{\mathcal{D}}(Y))$.

Theorem 6.1.3 (Greenlees) *For X and Y , $O(2)$ spectra with rational homotopy groups, there is a short exact sequence as below.*

$$0 \rightarrow \text{Ext}_{\mathcal{A}(\mathcal{D})}(\pi_*^{\mathcal{D}}(\Sigma X), \pi_*^{\mathcal{D}}(Y)) \rightarrow [X, Y]_*^{S_{\mathcal{D}}} \rightarrow \text{Hom}_{\mathcal{A}(\mathcal{D})}(\pi_*^{\mathcal{D}}(X), \pi_*^{\mathcal{D}}(Y)) \rightarrow 0$$

This follows since the homotopy category of $\mathcal{A}(\mathcal{D})$ clearly agrees with the algebraic model for dihedral spectra in [Gre98b]. We now introduce a construction that we will make much use of, we will soon see that this is an explicit description of the ‘global sections’ of an object of $\mathcal{A}(\mathcal{D})$.

Definition 6.1.4 *Let $N \geq 1$ and take $V \in \mathcal{A}(\mathcal{D})$. Then $\bigoplus_N V$ is defined as the following pullback in the category of $\text{Ch}(\mathbb{Q}[W])$ -modules.*

$$\begin{array}{ccc} \bigoplus_N V & \longrightarrow & \prod_{k \geq N} V_k \\ \downarrow & & \downarrow \\ V_{\infty} & \longrightarrow & \text{tails}(V) \end{array}$$

We also have $\bigoplus_N^W V := (\bigoplus_N V)^W$ the W -fixed points of $\bigoplus_N V$.

Note that $\text{tails}(V)^W = \text{colim}_n \prod_{k \geq n} (V_k^W)$ and that the structure map of V induces a map $V_{\infty} \rightarrow \text{tails}(V)^W$. So we can construct $\bigoplus_N^W V$ in terms of a pullback in $\text{Ch}(\mathbb{Q})$. The notation \bigoplus_N is to make the reader think of some combination of a direct product and a direct sum. Indeed if $V_{\infty} = 0$, then $\bigoplus_N V = \bigoplus_{k \geq n} V_k$.

Lemma 6.1.5 *The category $\mathcal{A}(\mathcal{D})$ contains all small limits and colimits.*

Proof Take some small diagram V^i of objects of $\mathcal{A}(\mathcal{D})$. Define $(\text{colim}_i V^i)_{\infty} = \text{colim}_i (V_{\infty}^i)$ and $(\text{colim}_i V^i)_k = \text{colim}_i (V_k^i)$. The map below induces (via the universal properties of colimits) a structure map for $\text{colim}_i V^i$.

$$V_{\infty}^i \longrightarrow \text{colim}_n \prod_{k \geq n} V_k^i \longrightarrow \text{colim}_n \prod_{k \geq n} \text{colim}_i V_k^i$$

Limits are harder to define because we are working with a stalk-based description of a ‘sheaf’, we will make this clear shortly. Let $(\lim_i V^i)_k = \lim_i (V_k^i)$ and $(\lim_i V^i)_{\infty} = \text{colim}_N \lim_i (\bigoplus_N^W V^i)$. The structure map is then as below.

$$(\lim_i V^i)_{\infty} = \text{colim}_N \lim_i \left(\bigoplus_N^W V^i \right) \longrightarrow \text{colim}_N \lim_i \prod_{k \geq N} V_k^i = \text{tails}(\lim_i V^i) \quad \blacksquare$$

We introduce a collection of useful adjunctions relating $\mathcal{A}(\mathcal{D})$ with rational chain complexes and W -equivariant rational chain complexes.

Definition 6.1.6 Let M be a rational chain complex, R in $\text{Ch}(\mathbb{Q}[W])$ and $V \in \mathcal{A}(\mathcal{D})$.

Define $i_k R$ to be the object of $\mathcal{A}(\mathcal{D})$ with $(i_k R)_\infty = 0$, $(i_k R)_n = 0$ for $n \neq k$ and $(i_k R)_k = R$. Now define p_k by setting $p_k V = V_k$, an object of $\text{Ch}(\mathbb{Q}[W])$. Then i_k is both right and left adjoint to p_k .

$$i_k : \text{Ch}(\mathbb{Q}[W]) \rightleftarrows \mathcal{A}(\mathcal{D}) : p_k \quad p_k : \mathcal{A}(\mathcal{D}) \rightleftarrows \text{Ch}(\mathbb{Q}[W]) : i_k$$

We can repeat this at infinity, though the inclusion i_∞ is now only a left adjoint. Let $p_\infty V = V_\infty$, a rational chain complex and define $(i_\infty M)_\infty = M$ and $(i_\infty M)_k = 0$. Then we have an adjunction

$$p_\infty : \mathcal{A}(\mathcal{D}) \rightleftarrows \text{Ch}(\mathbb{Q}) : i_\infty.$$

We set cM to be the object of $\mathcal{A}(\mathcal{D})$ with $cM_k = M = cM_\infty$ and structure map induced by the diagonal map $M \rightarrow \prod_{k \geq 1} M$. Then we have the ‘constant sheaf’ and ‘global sections’ adjunction.

$$c : \text{Ch}(\mathbb{Q}) \rightleftarrows \mathcal{A}(\mathcal{D}) : \bigoplus_1^W$$

It is time we made our analogy to sheaves clear. While a useful way to view the category, the sheaf description also provides the proof that $\mathcal{A}(\mathcal{D})$ is a closed monoidal category.

Definition 6.1.7 Let \mathcal{P} be the space $\mathcal{FO}(2)/O(2) \setminus \{SO(2)\}$ (\mathcal{P} for points) and let \mathcal{O} be the constant sheaf of \mathbb{Q} on \mathcal{P} , considered as a sheaf of rings. Now consider W -equivariant objects and W -equivariant maps in $\mathcal{O}\text{-mod}$, we denote this category by $W\mathcal{O}\text{-mod}$.

To specify an \mathcal{O} -module M one only needs to give the stalks at the points k and ∞ and a \mathbb{Q} map $M_\infty \rightarrow \text{tails}(M)$. The global sections of M are then given by $\bigoplus_1 M$. Hence any object of $\mathcal{A}(\mathcal{D})$ defines an object of $W\mathcal{O}\text{-mod}$. The inclusion functor has a right adjoint: fix . On an W -equivariant \mathcal{O} -module V , $\text{fix}(V)_k = V_k$, $\text{fix}(V)_\infty = V_\infty^W$ and the structure map is $V_\infty^W \rightarrow V_\infty \rightarrow \text{tails}(V)$. Hence we have an adjoint pair

$$\text{inc} : \mathcal{A}(\mathcal{D}) \rightleftarrows W\mathcal{O}\text{-mod} : \text{fix}$$

thus we can view $\mathcal{A}(\mathcal{D})$ as a full subcategory of $W\mathcal{O}\text{-mod}$. Our definitions and constructions are, therefore, slight adjustments to the usual definitions of modules over a sheaf of rings.

In particular, one could also describe the limit of some diagram V^i in $\mathcal{A}(\mathcal{D})$ as $\text{fix} \lim_i \text{inc} V^i$, where the limit on the right is taken in the category of $W\mathcal{O}$ -modules.

Lemma 6.1.8 The category $\mathcal{A}(\mathcal{D})$ is a closed symmetric monoidal category. Furthermore the adjunction (c, \bigoplus_1^W) is symmetric monoidal.

Proof The category of \mathcal{O} -modules is closed symmetric monoidal, hence so is the category $W\mathcal{O}\text{-mod}$. That is, for M and N in $W\mathcal{O}\text{-mod}$, W acts diagonally on $(M \otimes_{\mathcal{O}} N)(U)$ and by conjugation on $\text{Hom}_{\mathcal{O}}(M, N)(U)$ (for U an open subset of \mathcal{P}). We ‘restrict’ this structure to $\mathcal{A}(\mathcal{D})$. Take A, B and C in $\mathcal{A}(\mathcal{D})$, the tensor product $A \otimes_{\mathcal{O}} B$ is in $\mathcal{A}(\mathcal{D})$ and is given by: $(A \otimes_{\mathcal{O}} B)_k = A_k \otimes_{\mathbb{Q}} B_k$, $(A \otimes_{\mathcal{O}} B)_{\infty} = A_{\infty} \otimes_{\mathbb{Q}} B_{\infty}$ and the structure map is given by the composite of the three maps below.

$$\begin{aligned} A_{\infty} \otimes B_{\infty} &\longrightarrow \text{colim}_n (\prod_{k \geq n} A_k) \otimes \text{colim}_n (\prod_{k \geq n} B_k) \\ \text{colim}_n (\prod_{k \geq n} A_k \otimes \prod_{k \geq n} B_k) &\xrightarrow{\cong} \text{colim}_n (\prod_{k \geq n} A_k) \otimes \text{colim}_n (\prod_{k \geq n} B_k) \\ \text{colim}_n (\prod_{k \geq n} A_k \otimes \prod_{k \geq n} B_k) &\longrightarrow \text{colim}_n \prod_{k \geq n} (A_k \otimes B_k) \end{aligned}$$

We now have a series of natural isomorphisms, where we suppress notation for inc .

$$\begin{aligned} \mathcal{A}(\mathcal{D})(A \otimes_{\mathcal{O}} B, C) &= W\mathcal{O}\text{-mod}(A \otimes_{\mathcal{O}} B, C) \\ &\cong W\mathcal{O}\text{-mod}(A, \text{Hom}_{\mathcal{O}}(B, C)) \\ &\cong \mathcal{A}(\mathcal{D})(A, \text{fix Hom}_{\mathcal{O}}(B, C)) \end{aligned}$$

Thus we take the internal homomorphism object for $\mathcal{A}(\mathcal{D})$ to be $\text{fix Hom}_{\mathcal{O}}(B, C)$. It is routine to prove that (c, \biguplus_1^W) is a strong symmetric monoidal adjunction. \blacksquare

Our model structure is an alteration of the flat model structure from [Hov01a], noting that every sheaf on \mathcal{P} is automatically flasque so the fibrations are precisely the stalk-wise surjections. Let $I_{\mathbb{Q}}$ and $J_{\mathbb{Q}}$ denote the sets of generating cofibrations and acyclic cofibrations for the projective model structure on rational chain complexes, see [Hov99, Section 2.3]. Similarly we have $I_{\mathbb{Q}[W]}$ and $J_{\mathbb{Q}[W]}$ for $\text{Ch}(\mathbb{Q}[W])$.

Proposition 6.1.9 *Define a map f in $\mathcal{A}(\mathcal{D})$ to be a weak equivalence or fibration if f_{∞} and each f_k is a homology isomorphism or surjection. This defines a symmetric monoidal model structure on the category $\mathcal{A}(\mathcal{D})$. Furthermore this model structure is cofibrantly generated, proper and satisfies the monoid axiom. The generating cofibrations, I , are the collections $cI_{\mathbb{Q}}$ and $i_k I_{\mathbb{Q}[W]}$, for $k \geq 1$. The generating acyclic cofibrations, J , are $cJ_{\mathbb{Q}}$ and $i_k J_{\mathbb{Q}[W]}$, for $k \geq 1$.*

Proof Much of this proof is routine, so we only give brief details. Lemma 6.1.10 shows that \biguplus_1^W preserves filtered colimits, this deals with the smallness conditions on the generating sets.

As an example of the kind of argument necessary, we identify the maps with the right lifting property with respect to I . Let $f: A \rightarrow B$ be such a map, using the adjunctions of definition 6.1.6 it follows that each $f_k: A_k \rightarrow B_k$ must be a surjection and a homology isomorphism, as must $\biguplus_1^W f: \biguplus_1^W A \rightarrow \biguplus_1^W B$. In turn, each f_k^W is a homology isomorphism and a surjection so $\biguplus_n^W f$ is a surjection and a homology isomorphism for each $n \geq 1$. Taking colimits over n we see that f_{∞} is a surjection and homology isomorphism.

Left properness is immediate because colimits are defined stalk-wise. For right properness one must check what happens at infinity, following the same method as above.

The pushout product axiom is straightforward. The monoid axiom holds because the functors p_k and p_∞ are strong monoidal left adjoints such that if each $p_k f$ and $p_\infty f$ are weak equivalences then f is a weak equivalence in $\mathcal{A}(\mathcal{D})$. ■

Lemma 6.1.10 *The functors \bigoplus_N preserves filtered colimits for all $N \geq 1$.*

Proof This is a pleasant exercise, one checks injectivity and surjectivity by first dealing with the term at infinity, then dealing with the finite number of terms that aren't determined by that term. ■

Corollary 6.1.11 *The adjunctions of definition 6.1.6 are strong symmetric monoidal Quillen pairs.*

Lemma 6.1.12 *The collection $i_k \mathbb{Q}W$ for $k \geq 1$ and $c\mathbb{Q}$ are a set of compact, cofibrant and fibrant generators for this category.*

Proof If there are no maps from $i_k \mathbb{Q}W$ into an object V , then each V_k must be acyclic. If this holds, then the homology of $\bigoplus_1^W V$ is isomorphic to the homology of V_∞ . So if there are no maps from $c\mathbb{Q}$ into V , then $\bigoplus_1^W V$ and hence v_∞ are acyclic. Compactness is routine. ■

Remark 6.1.13 *The adjunction $\text{inc} : \mathcal{A}(\mathcal{D}) \rightleftarrows W\mathcal{O}\text{-mod} : \text{fix}$ can be used to put a new model structure on $W\mathcal{O}\text{-mod}$. Define a map f to be a weak equivalence or fibration if $\text{fix } f$ is, then we have a new, cofibrantly generated model structure on $W\mathcal{O}\text{-mod}$. With this model structure the adjunction (inc, fix) becomes a Quillen equivalence.*

6.2 Comparing $S_{\mathcal{D}}\text{-mod}$ and $\mathcal{A}(\mathcal{D})$

In this subsection we give the proof that $\mathcal{A}(\mathcal{D})$ and $S_{\mathcal{D}}\text{-mod}$ are monoidally Quillen equivalent. We are able to use same method of proof as [Bar09a] since for any two dihedral subgroups H and K , the set of maps

$$[O(2)/H_+ \wedge S_{\mathcal{D}}, O(2)/K_+ \wedge S_{\mathcal{D}}]_*^{S_{\mathcal{D}}}$$

is concentrated in degree zero. There are two key steps, the first uses [SS03b] and [Shi07] to replace $S_{\mathcal{D}}\text{-mod}$ by the category of rational Mackey functors. The second identifies $\mathcal{A}(\mathcal{D})$ with the category of rational Mackey functors using [Gre98b].

Recall that our model for dihedral spectra is the category of modules over the commutative ring spectrum $S_{\mathcal{D}}$ in the category of $O(2)$ -equivariant EKMM S -modules. The weak equivalences and fibrations of this model category are those maps which are weak equivalences or fibrations of underlying EKMM S -modules.

The ring spectrum $S_{\mathcal{D}}$ has rational homotopy groups and $\pi_*^H(S_{\mathcal{D}}) = 0$ for $H \in \mathcal{C}$ by [MM02, Theorem IV.6.13]. So it follows that any $S_{\mathcal{D}}$ -module X also has rational homotopy groups and $\pi_*^H(X) = 0$ for any $H \in \mathcal{C}$.

Lemma 6.2.1 *The model category $S_{\mathcal{D}}\text{-mod}$ is generated by the countably infinite collection of compact objects*

$$\mathcal{G} = \{S_{\mathcal{D}}\} \cup \{S_{\mathcal{D}} \wedge \widehat{ce}_H O(2)/H_+ \mid H \in \mathcal{D} \setminus \{O(2)\}\}$$

where \widehat{c} is the cofibrant replacement functor of $O(2)\mathcal{MS}$.

Proof We must prove that if X is an object of $S_{\mathcal{D}}\text{-mod}$ such that $[\sigma, X]_*^{S_{\mathcal{D}}} = 0$ (maps in the homotopy category of $S_{\mathcal{D}}\text{-mod}$) for all $\sigma \in \mathcal{G}$, then $X \rightarrow *$ is a π_* -isomorphism. As mentioned above $\pi_*^H(X) = 0$ for any $H \in \mathcal{C}$ and since $[S_{\mathcal{D}}, X]_*^{S_{\mathcal{D}}} = 0$ we see that $\pi_*^{O(2)}(X) = 0$. So now we must consider a finite dihedral group H : by [Gre98a, Example C(i)], $\pi_*^H(X)$ is given by $\bigoplus_{(K) \leq H} (e_K \pi_*^K(X))^{W_H K}$. We have assumed that $[S_{\mathcal{D}} \wedge \widehat{ce}_K O(2)/K_+, X]_*^{S_{\mathcal{D}}} = 0$, for each finite dihedral K , but this is precisely the condition that $e_K \pi_*^K(X) = 0$ for each K . Thus $\pi_*^H(X) = 0$ and our set generates the homotopy category. Compactness follows from the isomorphisms $[\sigma_H, X]_*^{S_{\mathcal{D}}} \cong e_H \pi_*^H(X)$. ■

Proposition 6.2.2 *The category of dihedral spectra is a spectral model category.*

Proof This follows from the fact that there is a strong monoidal Quillen pair

$$Sp_+^{\Sigma} \rightleftarrows S_{\mathcal{D}}\text{-mod}$$

as detailed just before Definition 5.13 in [Bar09a]. ■

Hence we can apply [SS03b, Theorem 3.3.3] to replace $S_{\mathcal{D}}\text{-mod}$ by a Quillen equivalent category of right modules over a spectral model category. But as with the finite case, we want to do so in a way that preserves the monoidal structure.

Definition 6.2.3 *The closure of \mathcal{G} under the smash product is called \mathcal{G}_{top} . Take the full subcategory of $S_{\mathcal{D}}\text{-mod}$ on object set \mathcal{G}_{top} , considered as a category enriched over symmetric spectra, we call this spectral category \mathcal{E}_{top} .*

*A **right module over \mathcal{E}_{top}** is a contravariant Sp_+^{Σ} -enriched functor $M: \mathcal{E}_{top} \rightarrow Sp_+^{\Sigma}$, the category of such functors and natural transformations is denoted $\text{mod-}\mathcal{E}_{top}$.*

The category of right modules over \mathcal{E}_{top} has a model structure with weak equivalences and fibrations defined objectwise in Sp_+^{Σ} , see [SS03b, Subsection 3.3] for more details. One can also define right modules using model categories other than Sp_+^{Σ} (such as $\text{Ch}(\mathbb{Q})$). Note that if we are talking about right modules over some \mathcal{C} -enriched category \mathcal{E} , then we mean contravariant \mathcal{C} -enriched functors from $\mathcal{E} \rightarrow \mathcal{C}$, with weak equivalences and fibrations defined using \mathcal{C} .

Proposition 6.2.4 *The adjunction below is a Quillen equivalence with strong symmetric monoidal left adjoint and lax symmetric monoidal right adjoint.*

$$(-) \otimes_{\mathcal{E}_{top}} \mathcal{G}_{top} : \text{mod-}\mathcal{E}_{top} \xleftrightarrow{\quad} S_{\mathcal{D}}\text{-mod} : \underline{\text{Hom}}(\mathcal{G}_{top}, -)$$

Proof That we have a Quillen equivalence is [SS03b, Theorem 3.9.3]. That the adjunction is strong monoidal is an exercise in manipulating coends, see [Bar08b, Theorem 9.1.2]. ■

Proposition 6.2.5 *There is a symmetric monoidal $\text{Ch}(\mathbb{Q})$ -enriched category \mathcal{E}_t , on object set \mathcal{G}_{top} , equipped with an isomorphism of monoidal graded \mathbb{Q} -categories:*

$$\pi_*(\mathcal{E}_{top}) \xrightarrow{\cong} H_* \mathcal{E}_t.$$

Such that there is a zig-zag of monoidal Quillen equivalences between $\text{mod-}\mathcal{E}_{top}$ (enriched over Sp_+^{Σ}) and $\text{mod-}\mathcal{E}_t$ (enriched over $\text{Ch}(\mathbb{Q})$).

Proof This is contained in the proof of [Bar09a, Theorem 6.5] which is based on [Shi07, Corollary 2.16]. ■

The model category $\text{mod-}\mathcal{E}_t$ is an algebraic model for dihedral spectra, albeit one which is not particularly explicit. We now need to relate $\text{mod-}\mathcal{E}_t$ to $\mathcal{A}(\mathcal{D})$.

Definition 6.2.6 *Let \mathcal{G}_a be the set of all tensor products of the objects $i_k \mathbb{Q}W$ for $k \geq 1$ and $c\mathbb{Q}$. Let \mathcal{E}_a denote the full subcategory of $\mathcal{A}(\mathcal{D})$ with objects set \mathcal{G}_a , considered as a category enriched over $\text{Ch}(\mathbb{Q})$.*

This result below follows immediately from [Bar09a, Theorem 4.9] and is an algebraic version of Proposition 6.2.4.

Proposition 6.2.7 *The adjunction below is a Quillen equivalence with strong symmetric monoidal left adjoint and lax symmetric monoidal right adjoint.*

$$(-) \otimes_{\mathcal{E}_a} \mathcal{G}_a : \text{mod-}\mathcal{E}_a \xleftrightarrow{\quad} \mathcal{A}(\mathcal{D}) : \underline{\text{Hom}}(\mathcal{G}_a, -)$$

We spend the rest of this subsection showing that \mathcal{E}_t and \mathcal{E}_a have Quillen equivalent categories of modules. To do so we find an isomorphism of categories enriched over graded \mathbb{Q} -modules between \mathcal{E}_a and $H_* \mathcal{E}_t \cong \pi_* \mathcal{E}_{top}$. We also want this isomorphism to be compatible with the symmetric monoidal structures of both categories. We follow the method of [Bar09a, Section 7].

By construction of $\pi_* \mathcal{E}_{top}$, there is a natural isomorphism

$$[\sigma, \sigma']_*^{S_{\mathcal{D}}} \longrightarrow \pi_* \mathcal{E}_{top}(\sigma, \sigma')$$

for any pair σ, σ' in \mathcal{G}_{top} . Similarly, there is a natural isomorphism as below, with the target a graded \mathbb{Q} -module concentrated in degree zero.

$$[\tau, \tau']_*^{\mathcal{A}(\mathcal{D})} \longrightarrow \mathcal{E}_a(\tau, \tau')$$

Both of these isomorphisms are compatible with the monoidal structures.

We will prove that the functor $\pi_*^{\mathcal{D}}$ gives a monoidal isomorphism from the full subcategory of $\text{Ho}(S_{\mathcal{D}})$ on object set \mathcal{G}_{top} to the full subcategory of $\text{Ho}(\mathcal{A}(\mathcal{D}))$ on object set \mathcal{G}_a . By the above, this will provide the desired monoidal isomorphism between $H_* \mathcal{E}_t$ and \mathcal{E}_a .

We have the following series of routine calculations, the first result tells us how to relate the set \mathcal{G}_{top} to \mathcal{G}_a . The second shows $\pi_*^{\mathcal{D}}$ is the isomorphism we need. The third is not necessary for the theory but gives an simple description of sets of maps between objects in \mathcal{G}_{top} .

Proposition 6.2.8 *For H a dihedral subgroup of $O(2)$ of order $2k$ and $i \geq 1$, let $\sigma_H^i = (S_{\mathcal{D}} \wedge \widehat{ce}_H O(2)/H_+)^i$, with the smash product taken in the category of $S_{\mathcal{D}}$ -modules. Then $\pi_*^{\mathcal{D}}(S_{\mathcal{D}}) = c\mathbb{Q}$ and $\pi_*^{\mathcal{D}}(\sigma_H^i) = i_k \mathbb{Q}[W]^{\otimes i}$ where the tensor product is the tensor product over \mathbb{Q} .*

Proposition 6.2.9 *Let $i, j, k, m \geq 1$ and let H and K be finite dihedral groups with $|H| = 2k$ and $|K| = 2m$. Then the functor $\pi_*^{\mathcal{D}}$ from Definition 6.1.2 gives isomorphisms as below.*

$$\begin{aligned} [S_{\mathcal{D}}, S_{\mathcal{D}}]_*^{S_{\mathcal{D}}} &\xrightarrow{\cong} [c\mathbb{Q}, c\mathbb{Q}]_*^{\mathcal{A}(\mathcal{D})} \\ [\sigma_H^i, S_{\mathcal{D}}]_*^{S_{\mathcal{D}}} &\xrightarrow{\cong} [i_k \mathbb{Q}W^{\otimes i}, c\mathbb{Q}]_*^{\mathcal{A}(\mathcal{D})} \\ [S_{\mathcal{D}}, \sigma_H^i]_*^{S_{\mathcal{D}}} &\xrightarrow{\cong} [c\mathbb{Q}, i_k \mathbb{Q}W^{\otimes i}]_*^{\mathcal{A}(\mathcal{D})} \\ [\sigma_H^j, \sigma_H^i]_*^{S_{\mathcal{D}}} &\xrightarrow{\cong} [i_k \mathbb{Q}W^{\otimes j}, i_k \mathbb{Q}W^{\otimes i}]_*^{\mathcal{A}(\mathcal{D})} \\ [\sigma_K^j, \sigma_H^i]_*^{S_{\mathcal{D}}} &\xrightarrow{\cong} [i_m \mathbb{Q}W^{\otimes j}, i_k \mathbb{Q}W^{\otimes i}]_*^{\mathcal{A}(\mathcal{D})} \end{aligned}$$

Proposition 6.2.10 *Let $i, j, k \geq 1$, then*

$$\begin{aligned} [c\mathbb{Q}, c\mathbb{Q}]_*^{\mathcal{A}(\mathcal{D})} &= e_{\mathcal{D}} A(O(2)) \\ [i_k \mathbb{Q}W^{\otimes i}, c\mathbb{Q}]_*^{\mathcal{A}(\mathcal{D})} &= \text{Hom}_{\mathbb{Q}}(\mathbb{Q}W^{\otimes i}, \mathbb{Q})^W \\ [c\mathbb{Q}, i_k \mathbb{Q}W^{\otimes i}]_*^{\mathcal{A}(\mathcal{D})} &= \text{Hom}_{\mathbb{Q}}(\mathbb{Q}, \mathbb{Q}W^{\otimes i})^W \\ [i_k \mathbb{Q}W^{\otimes j}, i_k \mathbb{Q}W^{\otimes i}]_*^{\mathcal{A}(\mathcal{D})} &= \text{Hom}_{\mathbb{Q}}(\mathbb{Q}W^{\otimes i}, \mathbb{Q}W^{\otimes j})^W \\ [i_n \mathbb{Q}W^{\otimes j}, i_k \mathbb{Q}W^{\otimes i}]_*^{\mathcal{A}(\mathcal{D})} &= 0. \end{aligned}$$

Proposition 6.2.11 *There is an isomorphism of monoidal $\text{Ch}(\mathbb{Q})$ -categories $H_* \mathcal{E}_t \cong \mathcal{E}_a$.*

Proof As mentioned above it suffices to give an isomorphism of monoidal categories enriched over graded \mathbb{Q} -modules (and hence of $\text{Ch}(\mathbb{Q})$ -categories with trivial differentials) $\pi_* \mathcal{E}_{top} \rightarrow \mathcal{E}_a$. Furthermore, we have a suitable enriched functor: $\pi_*^{\mathcal{D}}$ and our

calculations above show that this gives an isomorphism of enriched categories. This functor respects the monoidal structures since everything is concentrated in degree zero. ■

Now we apply [Bar09a, Theorem 7.5 and Corollary 7.6].

Lemma 6.2.12 *Let C_0 denote the (-1) -connective cover functor on $\text{Ch}(\mathbb{Q})$. There is a zig-zag of quasi-isomorphisms of monoidal $\text{Ch}(\mathbb{Q})$ -categories.*

$$\mathcal{E}_t \xleftarrow{\sim} C_0 \mathcal{E}_t \xrightarrow{\sim} H_* \mathcal{E}_t \cong \mathcal{E}_a$$

hence there is a zig-zag of monoidal Quillen equivalences of $\text{Ch}(\mathbb{Q})$ -model categories.

$$\text{mod-}\mathcal{E}_t \xleftarrow{\sim} \text{mod-}C_0 \mathcal{E}_t \xrightarrow{\sim} \text{mod-}H_* \mathcal{E}_t \cong \text{mod-}\mathcal{E}_a$$

Proof The map $C_0 \mathcal{E}_t \rightarrow \mathcal{E}_t$ is the inclusion and $C_0 \mathcal{E}_t \rightarrow H_* \mathcal{E}_t$ is the projection. These are quasi-isomorphisms because the homology of \mathcal{E}_t is concentrated in degree zero. That quasi-isomorphisms induce Quillen equivalences of module categories is [SS03b, Theorem A.1.1]. ■

Thus we have proven that the model category of dihedral spectra is Quillen equivalent to $\text{mod-}\mathcal{E}_a$ which we know is Quillen equivalent to $\mathcal{A}(\mathcal{D})$. Furthermore all Quillen equivalences involved are symmetric monoidal. Hence we have found a simple, algebraic category that contains all the homotopical information of dihedral spectra and is compatible with the monoidal structure.

Theorem 6.2.13 *The model categories, $S_{\mathcal{D}}\text{-mod}$ and $\mathcal{A}(\mathcal{D})$, are Quillen equivalent via symmetric monoidal Quillen pairs. Hence the homotopy categories of $S_{\mathcal{D}}\text{-mod}$ and $\mathcal{A}(\mathcal{D})$ are equivalent as symmetric monoidal categories.*

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